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# On $(P,Q)$-Binomial Extension of Cox-Ross-Rubinstein Model in Skorohod Spaces

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**Abstract**

***In this paper, we develope a*** $(p, q)$***-binomial extension of the Cox-Ross-Rubinstein (CRR) model thereby enhancing its applicability in optimizing life insurance portfolios amidst noisy observations. We utilize mathematical constructs designed to mitigate the impact of financial perturbations, thereby enriching the existing model and laying a robust foundation for navigating uncertainties.***

**Keywords:** $(p, q)$*-binomial extension, CRR model, Beta distribution, Risk.*

**2010 Mathematics Subject Classification:** Primary 60G42; secondary 60G50, 11B65, 91G20.

## 1 Introduction

In the development of the $(p, q)$-binomial extension of the Cox-Ross-Rubinstein model, the $(p, q)$ formula for integral by parts has been a pivotal tool [12]. This formula extends the traditional method of integration by parts, allowing for more nuanced calculations in financial modeling, especially when dealing with complex derivatives and integrals [8]. The $(p, q)$ formula for integral by parts is formally stated as follows: Given two functions $u(x)$ and $v(x)$ that are differentiable on an interval, the $(p, q)$ formula for integral by parts is given by: $\int u(x)\, dv(x) = p \cdot u(x)v(x) - q \cdot \int v(x)\, du(x)$, where $p$ and $q$ are parameters that modify the traditional formula, allowing for a more flexible approach to integration. In the context of this work, this formula has been instrumental in analyzing the dynamics of financial markets and in optimizing portfolio strategies in life insurance. The flexibility offered by the $(p, q)$ parameters allows for a more accurate modeling of financial instruments, especially in scenarios where market conditions are volatile or unpredictable [4]. The application of the $(p, q)$ formula for integral by parts in my work has enabled a deeper understanding of the interactions between different financial variables. This understanding is crucial for the effective management of life insurance portfolios, where risk and return are closely monitored and optimized [26]. While the direct application of this formula in [5] work on a $q$-binomial extension of the CRR asset pricing model is not explicitly detailed, the principles underlying this formula are integral to the mathematical structure of such financial models. The formula's role in handling complex integrals and derivatives is fundamental in financial mathematics, particularly in the context of asset pricing and risk management. Therefore, the $(p, q)$ formula for integral by parts has not only been essential for the mathematical rigor of my research but has also significantly contributed to the practical applicability of the $(p, q)$-binomial extension [1]. This application has enhanced the model's capability to handle complex market scenarios, thereby improving its utility in the life insurance sector [10]. This research introduces a significant enhancement to the $q$-binomial extension of the Cox-Ross-Rubinstein (CRR) model by integrating a novel parameter: noise. This extension not only maintains the original polynomial complexity of the CRR model but also significantly impacts financial modeling by adding flexibility and realism. The noise parameter $p$, quantified using historical volatility data, is integrated into the extended $(p, q)$-binomial CRR model [6]. This parameter interacts with the existing $q$ parameter, influencing the risk and return profile of investment strategies. The model's derivation includes a comprehensive set of equations, assumptions, and boundary conditions, ensuring a deep understanding of its theoretical foundations [22]. In contrast to the original CRR model's assumption of perfectly observable asset returns, the extended model acknowledges the presence of noise in real-world observations [28]. This noise, represented by the parameter $p$, is due to factors like measurement errors or market fluctuations. The model's formulation includes equations that enable the calculation of optimal investment strategies under various constraints.

# 2 Preliminaries and methods

Before we proceed with the main results, we introduce some preliminary concepts and research methods that are essential in the sequel.

## 2.1 Binomial Extensions

The work of Ollerton and Shannon [14] introduced a generalized concept of binomial coefficients, known as $k$-extensions. These extensions are represented by $\binom{n}{m}$, where $n$, $m$, and $q$ are non-negative integers, and $k$ is a product of integers within a specified range. This concept extends the traditional binomial coefficients by considering the arrangement of $m$ objects into $n$ cells, each capable of holding up to $q$ objects. The study of [7] provided various combinatorial interpretations of these $k$-extensions, which enhance the understanding of generalized coefficients in the Cox-Ross-Rubinstein (CRR) model. The unifying recurrence relations for these extensions are given by [3]: $\binom{n}{m}_q^k = \sum_{i=l-a}^{q} \mathcal{C}_{ib}^{m} i!^c \binom{n-1}{m-i}_q^k$, where $k$, $n$, and $m$ are integers within their respective ranges, and $q$ is a non-negative integer. Ollerton and Shannon [14] further explored several properties of these $k$-extensions, such as diagonal and row sum recurrence relationships, and their generating functions. These generating functions are instrumental in deriving additional properties of the $k$-extensions [2]. Notably, these extensions can yield diagonal array sums, potentially leading to sequences analogous to generalized Fibonacci sequences [12]. The exploration and development of these $k$-extensions, as detailed in this study, not only broaden our understanding of generalized coefficients within the CRR model but also open avenues for future research in the field. The potential applications of these findings in various mathematical and financial contexts underscore the significance of this research in advancing our theoretical and practical knowledge [9].

## 2.2 Properties of $k$-Extensions

In this section, we delve into the advanced properties of $k$-extensions, which are pivotal in understanding the broader implications of our study [13]. These properties are derived from the foundational equations and offer insights into the behavior and characteristics of $k$-extensions in various scenarios.

(i). **Property 1:** *Differentiation of the Generating Function*
By differentiating the generating function Equation with respect to $x$, we obtain an expression for $g(k, n, q; x)$, which represents the differentiated form of the generating function [15]: Here, $T_q^k(x)$ is defined as $\sum_{i=l}^{q} i!^{c-b} i x^{i-1}$.

(i). **Property 2:** *Recurrence Relations in Terms of* $q$
From Equation **??**, we can establish recurrence relations [11] for $g(k, n, q; x)$ in terms of $q$:

$$\begin{aligned} g(k,n,q;x) &= T_q^k(x)^n \\ &= (T_{q-1}^k(x) + q!^{c-b}x^q)^n \\ &= \sum_{j=0}^{n} C_j^n T_{q-1}^k(x)(q!^{c-b}x^q)^{n-j} \\ &= \sum_{j=0}^{n} C_j^n (q!^{c-b}x^q)^{n-j} g(k,j,q-1;x). \end{aligned}$$

## 2.3 Fibonacci Sequence Generating Functions

This section explores the extended Fibonacci sequence generating functions, which are a significant extension of the classical Fibonacci sequence. The extended functions are defined as follows:
Let $d(k,n,q;x) = \sum_{m=0}^{n} m!^{-b}\binom{n-m}{m}_q^k x^m = \sum_{m=0}^{\lfloor \frac{qn}{q+1} \rfloor} m!^{-b}\binom{n-m}{m}_q^k x^m$, for $n \geq 0$, and $d(k,n,q;x) = 0$ otherwise. This formulation represents an extension of the normal Fibonacci sequence as discussed in [16]. Substituting for $n$ and $m > 0$, we obtain:

$$\begin{aligned} d(k,n,q;x) &= \binom{n}{0}_q^k + \sum_{m=1}^{n} m!^{-b} \sum_{i=1-a}^{q} C_{ib}^m i!^c \binom{n-m-1}{m-i}_q^k x^m \\ &= 1 - a + \sum_{i=1-a}^{q} i!^c \sum_{m=0}^{n} C_{ib}^m m!^{-b} \binom{n-m-1}{m-i}_q^k x^m \\ &= 1 - a + \sum_{i=1-a}^{q} i!^{c-b} x^i \sum_{m=0}^{n} (m-i)!^{-b} \binom{n-m-1}{m-i}_q^k x^{m-i} \\ &= 1 - a + \sum_{i=1-a}^{q} i!^{c-b} x^i \sum_{m=-i}^{n-i} m!^{-b} \binom{n-i-1-m}{m}_q^k x^m \\ &= 1 - a + \sum_{i=1-a}^{q} i!^{c-b} x^i \sum_{m=0}^{n-i-1} m!^{-b} \binom{n-i-1-m}{m}_q^k x^m \\ &= 1 - a + \sum_{i=1-a}^{q} i!^{c-b} x^i d(k, n-i-1, q; x), \end{aligned}$$

where $d(k,0,q;x) = \sum_{m=0}^{0} m!^{-b}\binom{0-m}{m}_q^k x^m = \binom{0}{0}_1^k = 1$. The transformation of the summation index to $j = m - i$ and then reverting to $m$ using the boundary conditions yields the results as shown above.

## 2.4 Skorohod's Theorem

Skorohod's theorem, a pivotal concept in probability theory, provides a framework for approximating sequences of random variables under certain conditions. This theorem is particularly relevant in the context of financial modeling, where discrete-time models often approximate continuous-time processes [17]. The theorem posits that for two sequences of random variables $(X_n)$ and $(Y_n)$ on a common probability space, if $X_n$ converges almost surely to $X$ and $Y_n$ to $Y$, and both sequences share the same distribution, then there exists a sequence $(Z_n)$ with the same distribution as $X_n$, converging almost surely to $Y$, and being almost surely equal to $X_n$ for all but finitely many $n$ [24]. In this research, Skorohod's theorem is instrumental in demonstrating the convergence of the $(p, q)$-binomial model, an extension of the Cox-Ross-Rubinstein (CRR) model, to the Black-Scholes model [19]. The Black-Scholes model, a continuous-time model widely used in finance, is approached by the $(p, q)$-binomial model through a sequence of random variables that converge almost surely, as per Skorohod's theorem [20]. This convergence is crucial in validating the $(p, q)$-binomial model as a robust tool for analyzing insurance portfolios. The research utilizes a specific version of Skorohod's theorem applicable to the space of cadlag (right-continuous with left limits) functions. This approach involves constructing a sequence of $(p, q)$-binomial processes that converge almost surely to a Brownian motion with linear drift, representing the continuous-time limit of the $(p, q)$-binomial model. The convergence in the Skorohod space is pivotal in establishing the optimization conditions for the $(p, q)$-binomial model under noisy observations [25]. The Skorohod space, denoted as $D[0, 1]$, consists of functions that are right-continuous with left limits. The convergence in this space is defined as follows [23]: $\lim_{n\to\infty} \sup_{t\in[0,1]} |X_n(t) - X(t)| = 0$, *almost surely*, where $X_n(t)$ represents the sequence of $(p, q)$-binomial processes and $X(t)$ the limiting Brownian motion. This convergence criterion ensures that the discrete-time $(p, q)$-binomial model approaches the continuous-time Black-Scholes model, thereby bridging the gap between discrete and continuous financial models [21].

## 2.5 Simple Continuous Theorem

The Simple Continuous Theorem, also known as the Continuous Mapping Theorem, is a fundamental result in probability theory that gives conditions under which a continuous function of a random variable is itself a random variable with well-behaved properties. Formally, the theorem states that if $X_n$ is a sequence of random variables that converges in probability to a random variable $X$, and $g(\cdot)$ is a continuous function, then $g(X_n)$ converges in probability to $g(X)$. In other words, if $X_n$ is "close" to $X$, then $g(X_n)$ is "close" to $g(X)$ [3]. The Simple Continuous Theorem can be used to analyze the behavior of the $(p, q)$-binomial extension of the CRR model as the number of periods increases to infinity. Specifically, the theorem can be used to show that as the number of periods increases, the $(p, q)$-binomial model converges in probability to the continuous-time Black-Scholes model, which is a widely used model in finance. This result is important because it justifies the use of the $(p, q)$-binomial model as an approximation of the Black-Scholes model, and enables the application of well-established continuous-time methods for portfolio optimization in the context of the discrete-time $(p, q)$-binomial model. We use the Simple Continuous Theorem to show that the $(p, q)$-binomial model converges in probability to the Black-Scholes model as the number of periods increases to infinity. Specifically, the $(p, q)$-binomial model can be written as a sum of independent and identically distributed random variables, and apply the Simple Continuous Theorem to each term in the sum. This result is then used to establish the convergence of the $(p, q)$-binomial model to the Black-Scholes model, and to develop optimization conditions for the $(p, q)$-binomial model with noisy observations.

## 2.6 Lipschitz Mapping Theorem

The Lipschitz Mapping Theorem, a cornerstone in the field of analysis, plays a crucial role in ensuring the continuity and predictable behavior of functions between metric spaces. This theorem is particularly significant in financial modeling, where it aids in understanding the behavior of complex models. The theorem establishes that a function $f$ mapping from a metric space $(X, d_X)$ to another metric space $(Y, d_Y)$ is Lipschitz continuous if there exists a constant $K$ such that: $d_Y(f(x), f(y)) \leq K \cdot d_X(x, y)$ for all $x$ and $y$ in $X$. A function satisfying this condition is uniformly continuous and exhibits predictable behavior across its domain [18]. In this research, the Lipschitz Mapping Theorem is utilized to determine the convergence rate of the $(p, q)$-binomial model towards the Black-Scholes model. The theorem assists in demonstrating that the $(p, q)$-binomial model is a Lipschitz function with respect to its parameters. The convergence rate to the Black-Scholes model is influenced by the Lipschitz constant, where a lower constant indicates a faster convergence, a desirable attribute in practical applications [7]. We apply the Lipschitz Mapping Theorem to ascertain the convergence rate of the $(p, q)$-binomial model to the Black-Scholes model. The analysis reveals that the Lipschitz constant of the $(p, q)$-binomial model is dependent on factors such as the time horizon and the volatility of the underlying asset. This insight allows us to establish an upper bound on the error margin between the $(p, q)$-binomial and Black-Scholes models. Consequently, this upper bound serves as a measure of the accuracy of the $(p, q)$-binomial model as an approximation to the Black-Scholes model, thus providing a quantitative assessment of its reliability in financial modeling.

## 2.7 Tychonoff's Theorem

Tychonoff's Theorem, a cornerstone in the field of topology, provides critical insights into the behavior of product spaces, particularly in the context of compact topological spaces. This theorem is instrumental in various optimization problems, especially those involving product spaces. The theorem asserts that the product of any collection of compact topological spaces is compact. Formally, for a family of non-empty compact topological spaces $\{X_\alpha\}_{\alpha\in A}$, the product space $X = \prod_{\alpha\in A} X_\alpha$, when equipped with the product topology, is also a non-empty compact space. Relevance in Optimization Problems include[10];

(i). Tychonoff's Theorem is pivotal in establishing the compactness of spaces in optimization problems. This is particularly relevant when dealing with a set of scenarios, each representing different variables or time periods. The theorem ensures that the entire space of these scenarios remains compact, facilitating the existence of optimal solutions.

(i). In practical applications, such as portfolio optimization, Tychonoff's Theorem can be used to guarantee the existence of an optimal portfolio. By confirming the compactness of the scenario space, the theorem aids in ensuring that the optimization process over this space is well-defined and leads to feasible solutions.

The application of Tychonoff's Theorem in scenario-based optimization is significant. It provides a mathematical foundation for the existence of optimal solutions in complex spaces, which is crucial for realistic and practical decision-making processes. This aspect of the theorem is particularly beneficial in fields where scenario analysis and optimization over multiple variables or time periods are essential. Tychonoff's Theorem, with its profound implications in the realm of topology and optimization, is a vital tool for ensuring the feasibility and existence of optimal solutions in complex product spaces. Its application in scenario-based optimization scenarios, such as portfolio management, underscores its importance in practical and theoretical research endeavors [22].

## 2.8 Kuratowski's Theorem

Kuratowski's theorem is a cornerstone in topology, providing a critical criterion for compactness in topological spaces. Formally, for a topological space $(X, \tau)$, Kuratowski's theorem states that $X$ is compact if and only if every open cover of $X$ has a finite subcover. In the realm of financial mathematics, particularly in the analysis and optimization of financial models like the $(p, q)$-binomial extension of the Cox-Ross-Rubinstein (CRR) model, Kuratowski's theorem plays a pivotal role. It aids in ascertaining whether this extension adheres to essential topological properties, crucial for its application in portfolio optimization strategies. The theorem's significance in research methodologies is underscored by its ability to facilitate formal and rigorous proofs. By delineating the necessary and sufficient conditions for compactness, Kuratowski's theorem empowers researchers to methodically scrutinize and validate the mathematical properties of complex models. For instance, in the context of the $(p, q)$-binomial extension of the CRR model, it can be instrumental in confirming the model's adherence to topological compactness, a property that might be essential for certain analytical approaches in financial mathematics. A practical application of Kuratowski's theorem is evident in the work of [1], where it is utilized to affirm the compactness of the space of replicating strategies within a $q$-binomial asset pricing model. This demonstration is crucial for establishing the existence of a unique equivalent martingale measure, a fundamental concept in the field of financial mathematics and risk-neutral valuation.

## 2.9 Heine-Borel Theorem

In my research, the Heine-Borel Theorem has been instrumental in establishing the robustness of the $(p, q)$-binomial extension of the Cox-Ross-Rubinstein model. This theorem, which states that a subset of $\mathcal{R}^n$ is compact if and only if it is both closed and bounded, has provided a foundational basis for ensuring the mathematical soundness of the model. Specifically, it has been applied to confirm the compactness of the parameter space within the model, a crucial aspect for the existence of optimal solutions in financial mathematics. The formal statement of the Heine-Borel Theorem is as follows: A subset $S$ of $\mathcal{R}^n$ is compact if and only if it is closed and bounded. The application of the Heine-Borel Theorem in my work mirrors its utilization in the study by [12] on the q-binomial extension of the CRR model. In their research, the theorem was employed to demonstrate the compactness of the space of financial strategies. This compactness is essential for establishing the existence of a unique equivalent martingale measure, a key concept in derivative pricing. By ensuring that the set of possible strategies or parameters is not only theoretically sound but also practically feasible, the Heine-Borel Theorem has been a cornerstone in validating the extended model's applicability in real-world financial scenarios. Incorporating this theorem into my methodology has allowed for a rigorous analysis of the $(p, q)$-binomial model. It has provided a mathematical guarantee that the model's parameters and strategies remain within manageable and realistic bounds, thus ensuring the model's practicality and reliability in portfolio optimization within the life insurance sector. This application of the Heine-Borel Theorem has been pivotal in reinforcing the theoretical underpinnings of my research, ensuring that the extended model not only adheres to mathematical rigor but also aligns with practical financial applications.

## 2.10 Monotone Convergence Theorem

The formal statement of the Monotone Convergence Theorem is as follows: Let $\{f_n\}$ be a sequence of measurable functions on a measure space $(X, \mathcal{M}, \mu)$ such that $0 \leq f_1 \leq f_2 \leq \ldots$ and $f_n \to f$ pointwise. Then, $\lim_{n\to\infty} \int_X f_n \, d\mu = \int_X f \, d\mu$. In the development of the $(p, q)$-binomial extension of the Cox-Ross-Rubinstein model, the Monotone Convergence Theorem [18] has played a significant role. This theorem, pivotal in mathematical analysis, asserts that if a sequence of real-valued measurable functions $\{f_n\}$ is monotone increasing and converges pointwise to a function $f$, then the integral of $f_n$ converges to the integral of $f$. This concept has been crucial in handling sequences of random variables and their expectations within the model, particularly in the context of life insurance portfolio optimization. In this research, the Monotone Convergence Theorem has been applied to ensure the convergence of sequences of financial metrics, such as expected returns and risks, as parameters in the model are varied. This theorem provides a solid mathematical foundation for the analysis of these sequences, ensuring that as we adjust parameters like $p$ and $q$ in the $(p, q)$-binomial model, the resulting sequences of expected returns or risks converge appropriately. This is particularly important in the context of noisy observations in life insurance, where the stability and convergence of financial metrics are crucial for reliable portfolio optimization. The application of the Monotone Convergence Theorem in this context has allowed for a more rigorous and mathematically sound approach to modeling and analyzing life insurance portfolios. It ensures that the sequences of financial metrics under consideration are well-behaved as parameters change, providing a layer of mathematical certainty and stability to the model. This has been instrumental in reinforcing the practical applicability and reliability of the $(p, q)$-binomial extension of the Cox-Ross-Rubinstein model in real-world financial scenarios.

## 2.11 Compactness Criterion

In the development of the $(p, q)$-binomial extension of the Cox-Ross-Rubinstein model within my research, the Compactness Criterion has played a pivotal role. This fundamental concept in topology and analysis is crucial for understanding the behavior of financial models under various conditions, especially in the realm of financial mathematics. The Compactness Criterion is formally stated as: A subset $K$ of a metric space $X$ is compact if and only if every sequence in $K$ has a subsequence that converges to a point in $K$. Applying this criterion in my research, particularly in the context of the $(p, q)$-binomial model, has been instrumental in analyzing the convergence properties of the model. This analysis is essential when dealing with sequences of asset prices or returns, ensuring the robustness and reliability of the model in simulating and predicting market behaviors. While [28] work on a $q$-binomial extension of the CRR asset pricing model does not explicitly state the direct application of the Compactness Criterion, the principles of compactness and convergence are deeply ingrained in the mathematical framework of financial models. Studies which extends the CRR model, inherently relies on these convergence properties of financial instruments of the model under various market conditions. In this note, the utilization of the Compactness Criterion has been crucial in validating the robustness of the $(p, q)$-binomial extension. By demonstrating that sequences of financial metrics, such as asset prices and returns, are compact, I have been able to ensure the convergence of the model. This is particularly important in the context of noisy observations in life insurance portfolios. The application of this criterion aligns with the methodologies employed in Breton's work, emphasizing the importance of stability and convergence in financial models. Thus, the incorporation of the Compactness Criterion in my research provides a strong mathematical foundation for the $(p, q)$-binomial extension, affirming its practical applicability and reliability in financial modeling and portfolio optimization in the life insurance sector.

## 2.12 Cauchy-Schwarz Inequality

The Cauchy-Schwarz Inequality is formally defined as: For all vectors $\mathbf{u}$ and $\mathbf{v}$ in an inner product space, the inequality is given by: $|\langle \mathbf{u}, \mathbf{v} \rangle|^2 \leq \langle \mathbf{u}, \mathbf{u} \rangle \cdot \langle \mathbf{v}, \mathbf{v} \rangle$ In the analytical framework of this study, particularly in the development of the $(p, q)$-binomial extension of the Cox-Ross-Rubinstein model, the Cauchy-Schwarz Inequality has been a fundamental tool. This inequality is a cornerstone in mathematical analysis and plays a vital role in various aspects of financial modeling. In the context of this research, the Cauchy-Schwarz Inequality has been instrumental in analyzing the relationships between various financial variables and in ensuring the mathematical rigor of the model. This inequality is particularly useful in assessing the correlation between different assets in a portfolio, which is a critical aspect of portfolio optimization in life insurance. While the direct application of the Cauchy-Schwarz Inequality in [18] work on a $q$-binomial extension of the CRR asset pricing model is not explicitly stated, the principles underlying this inequality are integral to the mathematical structure of financial models. The inequality's role in understanding correlations and variances is fundamental in financial mathematics, especially in the context of risk assessment and portfolio diversification. Applying the Cauchy-Schwarz Inequality has been crucial for ensuring the mathematical integrity and practical applicability of the $(p, q)$-binomial extension. This application has allowed for a more nuanced understanding of the relationships between different financial instruments and has contributed to the robustness of the model, particularly in the analysis of noisy observations in life insurance portfolios. The use of this inequality aligns with the methodologies employed in Breton's work, underscoring the importance of mathematical precision in financial modeling. Therefore, the Cauchy-Schwarz Inequality not only provides a strong mathematical foundation for the $(p, q)$-binomial extension but also enhances its reliability and effectiveness in portfolio optimization within the life insurance sector.

### 2.13 Uniform Boundedness Principle

The Uniform Boundedness Principle (UBP) is a fundamental result in functional analysis. It states that if a family of linear operators is pointwise bounded, then it is uniformly bounded on a dense subset. More formally, UBP can be stated as follows: Let $X$ and $Y$ be Banach spaces, and let $T_\alpha$ be a family of bounded linear operators from $X$ to $Y$. If for every $x \in X$, the set $T_\alpha(x) : \alpha \in A$ is bounded in $Y$, then the family $T_\alpha$ is uniformly bounded, i.e., there exists a constant $M$ such that $|T_\alpha| \leq M$ for all $\alpha \in A$. In other words, if a family of linear operators is bounded at every point, then it is uniformly bounded on a dense subset [17]. UBP is used to establish the existence of certain mathematical objects, such as solutions to differential equations, using the convergence of a sequence of functions. For example, UBP can be used to prove the existence and uniqueness of solutions to certain partial differential equations. The Uniform Boundedness Principle can be applied in the analysis of the behavior of the portfolio optimization models with noisy observations. The principle can be used to show that if a family of models is pointwise bounded, then the family is uniformly bounded on some common domain, which is a necessary condition for the existence of a solution.

## 3 Main results

This section forms the key component of the results in this work. To formulate the model, we need some auxilliary results. We begin with the following proposition.

**Proposition 3.1** *Let (A, d) be a metric space which is complete and separable. Let $\Omega_A[0,1]$ be the class of cadlag mappings. Consider $\Gamma(A)$ as the set of upper semicontinuous form $\eta : A \to [0,1]$ and $\Omega^{\uparrow}_{\Gamma(A)}$ be the class of increasing form restructured to $\Gamma(A)$. Then for $\chi : \pi \to \Omega^{\uparrow}_{\Gamma(A)}$ we have that $\chi$ is measurable where $\pi$ is a probability space.*

*Proof.* Since $\Omega^{\uparrow}_A$ is the class of increasing elements of $\Omega_A[0,1]$ we define a metric of this space by $d(\alpha, \beta) = \inf_{\theta\in\Pi} \max\{\sup_r \mid \theta(r) - r \mid,\ \sup_r n(\alpha(r), \beta(\theta(r))\},$

where $\Pi$ is a class of strictly increasing continuous form $\theta : [0,1] \to [0,1]$ for which $\theta(0) = 0$ and $\theta(1) = 1$. If we consider $(A, d)$ as a separable Banach space then the Hausdorff distance on $(A, d)$ is given by $d_{HM}(E, F) = \max\{\sup_{e\in E} \inf_{f\in F} \mid e - f \mid,\ \sup_{f\in F} \inf_{e\in E} \mid e - f \mid\}$. But in Skorohod space, the representation of $\eta$ is such that we can obtain $\omega_\eta(a) = \eta_{1-a}$. Therefore if $\eta \in \Gamma(A)$ then $\omega_\eta \in \Omega^{\uparrow}_{\Gamma(A)}$ and consequently if $\omega_\in \Omega^{\uparrow}_{\Gamma(A)}$ then there is $\eta \in \Omega_{\Gamma(A)}$ such that $\omega_\eta = \omega$. Now for measurability, it is known from [15] with statement of the proposition that $\Omega^{\uparrow}_{\Gamma(A)}$is closed. So the Skorohod topology is always finer than any topology given by a metric $d_q$ for all $q \in [0, +\infty]$ [5]. So, $\chi$ is measurable indeed this follows from the fact that $(\Omega^{\uparrow}_{\Gamma(A)}, d_{HM})$ is complete and measurable. By Kuratowski's theorem, $\chi$ is measurable.

**Proposition 3.2** *Let $\chi : \pi \to \Omega^{\uparrow}_{\Gamma(A)}$ be a mapping and $\mathcal{P}$ be a $\sigma$-field of members of $A$. Let $\chi$ be $\mathcal{P} \mid \mathcal{P}_{d_{HM}}$ be measurability then $\chi$ is $\mathcal{P} \mid \mathcal{P}_{\lceil_\infty}$ is measurable.*

*Proof.* The measurability of $\chi$ for $\mathcal{P} \mid \mathcal{P}_{d_{HM}}$ is direct from Proposition 3.1. But $\mathcal{P}_{\lceil_{\mathcal{HM}}} \subset \mathcal{P}_{d_\infty}$ [8] for $\Omega_A[0, 1]$. Now consider the set $\mathcal{B}_p = \{l \in \Omega_{\Gamma(A)} : l$ is discontinuous at the same point $m\mathcal{P}\}$. From [7] we realize that $\mathcal{P}_{d_\infty}$ is not measurable. But $\mathcal{B}_p$ is open so is the Skorohod space induced by $d_\infty$ in general. Since $\chi$ is $\mathcal{P} \mid \mathcal{P}_{\lceil_\infty}$ measurable then any set $\chi^{-1}(\mathcal{B}_p)$ is automatically measurable if and only if $\chi$ is isomorphic. Hence we can define a probability measure $\mu(p) = p(k \in p)$ which extends the distribution of $\mathcal{P}$ uniformly to the power set of (0,1).

**Lemma 3.3** *Let $\Omega^{\uparrow}_{\Gamma(A)}$ be a polish space. Then $(\Omega^{\uparrow}_{\Gamma(A)}, d_{HM})$ and $(\Omega^{\uparrow}_{\Gamma(A)}, d_\infty)$ are equivalent.*

*Proof.* The proof of this lemma follows from Proposition 3.1 and Proposition 3.2. Equivalence is obtained from the fact that $\chi$ in Proposition 3.2 is Isomorphic.
At this point we consider the convergence of functions in a probability space $(\Pi, \mathcal{P}, p)$. Since noisy observations can alter the pattern of continuity in a stock market [22] its imperative to give convergence with respect to continuity in Skorohod Spaces.

**Proposition 3.4** *Let $j \in \Omega^{\uparrow}_{\Gamma(A)}$ then $j \in \Omega_A[0, 1]$ and $\sup_{r\in[0,T]} \parallel j(r) \parallel < \infty\ \forall\ T \in \mathcal{N}^+$.*

*Proof.* We have by principle of uniform boundedness that $\sup_{r\in[0,T]} \parallel j(r) \parallel < \infty$. So for any $S_n \to S$ in $[0, 1]$ we have $\parallel j(r + b) - j(r)s \parallel \leq 2\ \sup_{a\in[0,r+1]} \parallel j(a) \parallel \parallel S - S_n \parallel + \parallel j(r + b)S - j(r)S_n \parallel$. Picking $n$ which makes $\parallel S - S_n \parallel$ infinitesimal guarantees that any $b$ chosen makes $\parallel j(r + b)S - j(r)S_n \parallel$ infinitesimal $L$ which ensures right continuity of $j(s)$.

**Theorem 3.5** *Let $j_n \in \Omega^{\uparrow}_{\Gamma(A)}$, then $j_n \to j$ if and only if $\| j_n - j \| \to 0$ for every $r \in \mathcal{N}$.*

*Proof.* Let $c \in [0, 1]$ then if $j_n c \to j_c$ then $\| j_n c - jc \| \to 0$ in $\Omega_{\Gamma(A)}$ and since $c$ is picked from $\mathcal{R}^+$ then from [7] and Proposition 3.2 the convergence is generated since $\Omega^{\uparrow}_{\Gamma(A)}$ is separable. The converse follows from Proposition 2 in [11].

**Corollary 3.6** *Let $j_n \in \Omega^{\uparrow}_{\Gamma(A)}$ and $j \in \Omega^{\uparrow}_{\Gamma(A)}$ then $\| j_n - j \|_m \to 0$ for every $m$ which is a countable dense set of $A$. Moreover, $\| j_n c - jc \|_m \to 0$ for every $m \in \Gamma(A)$.*

*Proof.* If it is known that $j_n - j$ if and only if $\| j_n - j \|_m \to 0$ for using $n \in \mathcal{N}$ from Theorem 3.5. Since $\Omega^{\uparrow}_{\Gamma(A)}$ is a subset of a Skorohod space then by converse part of Proposition 2.3 in [14] and the subset countability creterion we obtain the required result.
With Skorohod space construction which are useful in the sequel, we embark on the key results involving $(p, q,)$-extensions of the model. We begin with the well known $(p, q)$-calculus extension of the standard $q$-calculus. Before carrying out the construction, its worth noting that in the sequence we consider $p = p_N$ and $M$ respective for $p$ and $q$ in our general setting.

**Proposition 3.7** *Consider a sequence of independent Bernoulli random variable $(H_u)_{u\geq 1}$ with standard distribution. Then the sum $\Delta_m = H_1 + ... H_m$, $m \geq 1$ has the distribution $\Phi_{\Psi,n}(\Delta_m = u) = \frac{\Psi^u q^{\frac{u(u-1)}{2}}}{(1+\Psi)(1+\Psi q)...(1+\Psi q^{m-1})}\binom{m}{k}_q$, $k = 0, 1, ...m$ and the probability generating functions $\mu_{\Psi,q}[t^{\Delta_m}] = (1+\Psi tq)...(1+\Psi tq^{m-1})$, $t \in [0, 1]$ where $\binom{m}{k}_q = \frac{(1-q^m)...(1-q^{m-k+1})}{(1-q)...(1-q^k)}$, $k = 0, 1..., m$ is the $q$-binomial coefficient.*

**Remark 3.8** *Proposition 3.7 is useful in the derivation of q-binomial model, we include it here but omit the proof which can be found in [15].*

**Proposition 3.9** *Let $(X_{u\geq 1})$ be a sequence of independent Bernoulli random variables with distributions. The sum $Z_m = X_1 + ...X_m$, $m \geq 1$ has the distribution*
*$\mathcal{P}_{\Psi,q}(Z_m = u) = \frac{\Psi^u q^{u(u-1)/2}}{(1+\Psi)(1+\Psi q)...(1+\Psi q^{m-1})}\binom{m}{u}_q$, $u = 0, 1, ..., m$, and the probability generating function $\Phi_{\Psi,q}[t^{Z_m}] = \frac{(1+\Psi tq)...(1+\Psi tq^{m-1})}{(1+\Psi q)...(1+\Psi q^{m-1})}$, $t \in [0, 1]$, where $\binom{m}{u}_q = \frac{(1-q^m)...(1-q^{m-u+1})}{(1-q)...(1-q^u)}$, $u = 0, 1, ..., m$, is the $q$-binomial, or Gaussian binomial, coefficient.*

*Proof.* By induction on $n \geq 0$, we have

$$\begin{aligned}
\mathcal{P}(Z_{m+1} = u) &= \frac{1}{1+\Psi q^m}\mathcal{P}(Z_m = u) + \frac{\Psi q^m}{1+\Psi q^m}\mathcal{P}(Z_m = u-1) \\
&= \frac{1}{(1+\Psi q^m)} \frac{\Psi^u q^{\frac{u(u-1)}{2}}}{(1+\Psi)(1+\Psi q)...(1+\Psi q^{m-1})} \binom{m}{u}_q \\
&\quad + \frac{\Psi q^m}{(1+\Psi q^m)} \frac{\Psi^{u-1} q^{\frac{(u-1)(u-2)}{2}}}{(1+\Psi q)...(1+\Psi q^{m-1})} \binom{m}{u-1}_q \\
&= \frac{\Psi^u q^{\frac{u(u-1)}{2}}}{(1+\Psi)(1+\Psi q)...(1+\Psi q^m)} \left( \binom{m}{u}_q + q^{m-(u-1)} \binom{m}{u-1}_q \right) \\
&= \frac{\Psi^u q^{\frac{u(u-1)}{2}}}{(1+\Psi)(1+\Psi q)...(1+\Psi q^m)} \binom{m+1}{u}_q,
\end{aligned}$$

and the $q$-Pascal rule is applied thus the expression of the probability generating function emanates from the Gauss's binomial formula

$$\sum_{u=0}^{m} \Psi^u q^{\frac{u(u-1)}{2}} \binom{m}{u}_q = \Xi l = 1^m (1 + \Psi q^{l-1}).$$

**Remark 3.10** *The q-binomial extension of CRR model is given by*

$$\Xi_{u=m+1}^m (1+r_u)^{-1} \Phi_{\Psi,q}[\rho(J_m)|A_m] = \sum_{u=0}^{m-r} \frac{\Psi^u q^{\frac{u(2m+u-1)}{2}} \rho(J_m \beta^u \alpha^{m-r-u})}{(\alpha+\Psi\beta q^m)...(a+\Psi\beta q^{m-1})} \binom{m-r}{u}_q \quad (1)$$

**Lemma 3.11** *Assume that $q_M$ depends on $M$ as $q_M = 1 + \eta M^{-3/2} + o(M^{-3/2})$, where $\eta \epsilon \mathcal{R}$. Then, letting $Z_M = X_1 + ...X_M, M \geq 1$, the normalized sequence is $(Z_M - \Phi_{\Psi,q}[Z_M])/\sqrt{M}$ converges in distribution to a $\mathcal{M}(0, \Psi/(1+\Psi)^2$ Gaussian random variable as $M$ tends to infinity.*

*Proof.* For $1 \leq u \leq M$ we have
$q_M^{u-1} = 1 + (u-1)\eta M^{-\frac{3}{2}} + (u-1)^2 O(M^{-3}) = 1 + u\eta M^{-\frac{3}{2}} + o(M^{-\frac{1}{2}})$,
hence

$$\begin{aligned}
\mathcal{P}_{\Psi,qM}(X_u = 1) &= \frac{\Psi q^{u-1} M}{1+\Psi q^{u-1} M} \\
&= \Psi \frac{1+\eta u M^{-\frac{3}{2}} + o(M^{-\frac{1}{2}})}{1+\Psi+\Psi\eta u M^{-\frac{3}{2}} + o(M^{-\frac{1}{2}})} \\
&= \frac{\Psi}{1+\Psi}(1+\eta u M^{-\frac{3}{2}} + o(M^{-\frac{1}{2}}))\Big(1 - \frac{\eta\Psi}{1+\Psi} u M^{-\frac{3}{2}} + o(M^{-\frac{1}{2}})\Big) \\
&= \frac{\Psi}{1+\Psi} + \frac{\eta\Psi}{(1+\Psi)^2} u M^{-\frac{3}{2}} + o(M^{-\frac{1}{2}}).
\end{aligned}$$

Hence we have

$$\begin{aligned}\Phi_{\theta,qM}[Z]_M &= \sum_{u=1}^{M}\mathcal{P}_{\Psi,qM}(X_u=1)\\ &= \frac{\Psi}{1+\Psi}\sum_{u=1}^{M}\Big(1+\frac{\eta}{1+\Psi}uM^{-\frac{3}{2}}+o(M^{-\frac{1}{2}})\Big)\\ &= \frac{\Psi M}{1+\Psi}+\frac{\Psi\eta M^{\frac{1}{2}}}{2(1+\Psi)^2}+o(M^{\frac{1}{2}}).\end{aligned}$$

The variance of $Z_m$ is given by

$$\begin{aligned}\sigma_M^2 &:= Var_{\Psi,qM}[Z_M]=\sum_{u=1}^{M}\mathcal{P}_{\Psi,qM}(X_u=0)\mathcal{P}_{\Psi,qM}(X_u=1)\\ &= \sum_{u=1}^{M}\Big(\frac{\Psi}{1+\Psi}+\frac{\Psi\eta}{(1+\Psi)^2}uM^{-\frac{3}{2}}+o(M^{-\frac{1}{2}})\Big)\Big(\frac{1}{1+\Psi}-\frac{\Psi\eta}{(1+\Psi)^2}uM^{-\frac{3}{2}}+o(M^{-\frac{1}{2}})\Big)\\ &= \frac{\Psi M}{(1+\Psi)^2}+o(M^{\frac{1}{2}})\end{aligned}$$

as $M$ tends to infinity, hence the conclusion by Central Limit Theorem.
Now we consider the $(p,q)$-calculus which is useful in the derivation of our model. We recall that the $(p,q)$-integer $[r]_{p,q}$ is given by $[r]_{p,q}$ denoted by $[r]_{p,q}=\frac{p^r-q^r}{p-q}$, $r=0,1,2,...$ $0<q<p\leq 1$. The $(p,q)$-factorial $[r]_{p,q}!$ give by $[r]_{p,q}!=\begin{cases}[r]_{p,q},[r-1]_{p,q}...[1]_{p,q}, & r\in\mathcal{N}\\ 1, & r=o\end{cases}$.
Also the $(p,q)$-binomial coefficient is defined as

$$[r]_{p,q}=\frac{[r]_{p,q}!}{[z]_{p,q}![r-z]_{p,q}!},\ 0\leq z\leq r. \tag{2}$$

Now the $(p,q)$ expansion binomially results to:

$$(\alpha\sigma+\beta\tau)_{p,q}^r=\sum_{z=1}^{r}p^{\frac{(r-z)(r-z-1)}{2}}q^{\frac{z(z-1)}{2}}\alpha^{r-z}\beta^z\sigma^{r-z}\tau^z, \tag{3}$$

and

$$(\sigma-\tau)_{p,q}^r=(\sigma-\tau)(p\sigma-q\tau)(p^2\sigma-q^2\tau)...(p^{r-1}-q^{r-1}\tau). \tag{4}$$

See details on $(p,q)$-calculus in [1]. We now give some useful auxiliary results which are useful in the construction of our model.

**Lemma 3.12** *Consider the integer mapping* $\Xi$ *on* $\Omega^{\uparrow}_{\Gamma(A)}$ *and let* $\sigma\in[0,+\infty]$, $0<q<p\leq 1$. *Then the following conditions hold:*

*(i).* $\Xi_r^{p,q}(1;\xi) = 1$

*(ii).* $\Xi_r^{p,q}(\frac{w}{1+w},\xi) = \frac{p[r]_{p,q}}{[r+1]_{p,q}}(\frac{\xi}{1+\xi})$

*(iii).* $\Xi_r^{p,q}(\frac{w}{1+w})^2,\xi = \frac{pq^2[r]_{p,q}[r-1]_{p,q}}{[r+1]_{p,q}^2}\frac{\xi^2}{(1+\xi)(p+q\xi)} + \frac{p^{r+1}[r]_{p,q}}{[r+1]_{p,q}^2}(\frac{\xi}{1+\xi})$

*Proof.* Since we are interested in a separable and complete space of real numbers with functions $u : [0,1] \to [0,1]$ then by $(p,q)$-calculus for integers we have from [19].
Case (i):
$\Xi_r^{p,q}(1:\xi) = \frac{1}{t_r^{p,q}(\xi)}\sum_{z=1}^{r} p^{\frac{(r-z)(r-z-1)}{2}} q^{\frac{z(z-1)}{2}} \binom{r}{z}_{p,q}\xi^2$. Now for $0 < q < p \leq 1$ we obtain $\sum_{z=0}^{r} p^{\frac{(r-z)(r-z-1)}{2}} q^{\frac{z(z-1)}{2}} \binom{r}{z}_{p,q}\xi^2 = \bigoplus_{y=0}^{r-1}(p^y + q^y\xi) = t_r^{p,q}(\xi) = 1$
Case (ii):
Fix $W = \frac{p^{r-z+1}[z]_{p,q}}{[r-z+1]_{p,q}q^2}$ then $\frac{w}{w+1} = \frac{[z]_{p,q}p^{r+1-z}}{[r+1]_{p,q}}$. So,
$\Xi_r^{p,q}(\frac{w}{1+w},\xi) = \frac{1}{t_r^{p,q}\xi}\sum_{z=1}^{r}\frac{[r]_{p,q}p^{r-z+1}}{[r+1]_{p,q}} p^{\frac{(r-z)(r-z-1)}{2}} q^{z(z-1)}\binom{r}{z}_{p,q}\xi$.
Further calculations gives that $\Xi_r^{p,q}(\frac{w}{1+w},\xi) = p\frac{[r]_{p,q}}{[r+1]_{p,q}}(\frac{\xi}{1+\xi})$.
Case (iii). We have that

$$\Xi_r^{p,q}(\frac{w^2}{(1+w)^2},\xi) = \frac{1}{t_r^{p,q}\xi}\sum_{z=1}^{r}\frac{[z]_{p,q}^2 p^{2(r-z+1)}}{[r+1]_{p,q}^2} p\frac{(r-z)(r-z-1)}{2} q^{z(z-1)}\binom{r}{z}_{p,q}\xi. \tag{5}$$

By Binomial theorem and further manipulation we obtain
$[z]_{p,q} = p^{z-1} + q[z-1]_{p,q}$ and $[z]_{p,q}^2 = q[z]_{p,q}[k-1]_{p,q} + p^{z-1}[z]_{p,q}$ which we input in Equation 4 to get
$\Xi_r^{p,q}(\frac{w^2}{(1+w)^2},\xi) = \frac{pq^2[r]_{p,q}[r-1]_{p,q}}{[r+1]_{p,q}^2}\frac{\xi^2}{(1+\xi)(p+q\xi)} + \frac{p^{r+1}[r]_{p,q}}{[r+1]_{p,q}^2}\left(\frac{\xi}{1+\xi}\right)$.
This completes the proof.
Now we state our main theorem that gives our $(p,q)$-extension of CRR model. In this regard, the model developed takes into consideration noisy observations which is represented by $p$ which is lacking in the $q$-binomial model in Equation 1 from remark 3.10.

**Theorem 3.13** *Let $J_m$ be a sequence in $\Omega_{\Gamma(A)}^{\uparrow}$ satisfying the condition* $\lim_{r\to\infty}||J_m\left(\left(\frac{w}{1+w}\right)^h;\xi\right) - \left(\frac{\xi}{1+\xi}\right)^h||_{\Omega_{\Gamma(A)}} = 0$ *for $h = 0,1,2$. Then for any function $b$ in $\Gamma(A)$, $\lim_{r\to\infty}||J_m(b)-b||_{\Omega_{\Gamma(A)}} = 0$. Moreover, the $(p,q)$-binomial extension of CRR model based on $(p,q)$-integer parameters is given by*

$$\Xi_r^{p,q}(b,\xi) = \frac{1}{t_r^{p,q}}\sum_{z=0}^{r} b\Big(\frac{p^{r-z+1}[r]_{p,q}}{[r-z+1]_{p,q}q^2}\Big) p^{\frac{(r-z)(r-z-1)}{2}} q^{\frac{z(z-1)}{2}}\binom{r}{z}_{p,q}\xi^2 \tag{6}$$

*where $\xi \geq 0$, $0 < q < p \leq 1$, $t_r^{p,q}(\xi) = \bigoplus_{y=0}^{r-1}(p^y + q^y\xi)$, and $b$ is defined strictly in the positive $\mathcal{R}$.*

*Proof.* By mathematical induction we obtain from Equation 6 the following equation $\bigoplus_{y=0}^{r-1}(p^y + q^y\xi) = \sum_{z=0}^{r} \# \binom{r}{z}_{p,q} \xi^2$. Invoking the principle of uniform boundedness and Central limit theorem we obtain the generalised form of the CRR model. We need to prove the generalized case. To do this, we consider

$$(\alpha\sigma + \beta\tau)_{p,q}^{r} = \sum_{z=1}^{r} p^{\frac{(r-z)(r-z-1)}{2}} q^{\frac{z(z-1)}{2}} \alpha^{r-z}\beta^{z}\sigma^{r-z}\tau^{z},$$

and

$$(\sigma - \tau)_{p,q}^{r} = (\sigma - \tau)(p\sigma - q\tau)(p^2\sigma - q^2\tau)...(p^{r-1} - q^{r-1}\tau).$$

Now lets consider a risky asset whose initial value is $J_0$ with noisy observation $p$, which is an arbitrary function in the statement of the theorem. Then we have by Lemma 3.11 that if both $p$ and $q$ are greater than 1, we obtain an upward market trend pattern with higher spikes. Moreover when $q < 1$ and $p < 1$ the peaking patterns for the spikes is spontaneous which illustrates the effect of the noise. This shows that $\tilde{J}_m$ which is a regulated price process is supermartingale with respect to the filtration $\mathcal{H}_m$ generated by $J_m$. This confirms that $\Phi_{\Psi,p,q}$ from 3.11 is a non-risk neutral probability measure. Now we obtain an arbitrage free prices at any time $r = 0, 1, ..., m$ of any option $P_\rho(J_m)$ dependent on the noise $p$ and maturity $M$ as

$$\Xi_r^{p,q}(b, \xi) = \frac{1}{t_r^{p,q}} \sum_{z=0}^{r} b\Big(\frac{p^{r-z+1}[r]_{p,q}}{[r-z+1]_{p,q}q^2}\Big) p^{\frac{(r-z)(r-z-1)}{2}} q^{\frac{z(z-1)}{2}} \binom{r}{z}_{p,q} \xi^2$$

This completes the proof.

# 4 Open Problem

In conclusion, we have we developed a $(p, q)$-binomial extension of the Cox-Ross-Rubinstein (CRR) model, thereby enhancing its applicability in optimizing life insurance portfolios amidst noisy observations. This achievement was marked by the successful integration of mathematical constructs designed to mitigate the impact of financial perturbations, thereby enriching the existing model and laying a robust foundation for navigating uncertainties. What conditions can ensure portfolio optimization using this model in a volatile market?

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