

REACHABILITY IN COMPLETE t -ARY TREES

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ABSTRACT. Mathematical trees such as Cayley trees, plane trees, binary trees, noncrossing trees, t -ary trees among others have been studied extensively. Reachability of vertices as a statistic has been studied in Cayley trees, plane trees, noncrossing trees and recently in t -ary trees where all edges are oriented from vertices of lower label towards vertices of higher label. In this paper, we obtain closed formulas as well as asymptotic formulas for the number of complete t -ary trees in which there are paths of a given length such that the terminal vertex is a sink, leaf sink, first child and non-first child. We also obtain number of trees in which there is a leftmost path of a given length.

1. INTRODUCTION

Plane trees (or *ordered trees*) are rooted trees whose children are ordered and are one of the structures counted by the famous Catalan numbers. Plane trees have been enumerated using a number of statistics such as number of vertices [2], number of leaves [2], degree sequences [3] among many others. A t -ary tree is a plane tree in which all the internal vertices have at most t children. If all the internal vertices have exactly t children then such a plane tree is called a *complete t -ary tree*. These are the main structures of our investigation. The total number of t -ary trees on n vertices or complete t -ary trees with n internal vertices is given by

$$\frac{1}{(t-1)(n-1)+1} \binom{t(n-1)}{n-1},$$

where t is a positive integer.

Given a vertex u of a plane tree, a vertex at a lower level which is incident to u , is said to be a *child* of u . The vertex at a higher level which is incident to u is the *parent* of u . The vertices with the same parent are called *siblings*. Since the siblings are linearly ordered, they are always drawn in a left-to-right pattern where the leftmost sibling is referred to as *first child*. At a given level, the leftmost child is the *eldest child*. A *leftmost path* refers to a sequence of edges joining eldest children at each level in a plane tree.

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In 2010, Du and Yin [4] introduced Cayley trees whose edges are oriented from vertices of lower label towards vertices of higher label. They called this orientation as *local orientation*. The study was initiated by an earlier work of Ethan Cotterill [1]. The study of graphs whose edges are oriented was unified by Remmel and Williamson in their study of digraphs [12]. *Indegree* (resp. *outdegree*) of a vertex v refers to the number of edges that are directed into (resp. away from) v in a directed graph. A *sink* (resp. *source*) is a vertex with outdegree (resp. indegree) 0. A *leaf sink* is a sink with indegree 1. A vertex i is *reachable* from a vertex j if there is a sequence of oriented edges from j to i . The number of edges on the path is the *length* of the path. These definitions were given in [8].

In his PhD thesis [8], Okoth enumerated Cayley trees whose edges are locally oriented. The statistics used by Okoth are number of vertices, path length and exact number of reachable vertices. Besides reachability, Okoth obtained a functional equation satisfied by the generating function of Cayley trees with a given number of sources and sinks. On reachability, the proofs were based on the decomposition of trees such that the vertices on a reachable path are roots of forests. Okoth obtained, mostly closed formulas and in some instances he obtained asymptotic formulae. The aforementioned author extended the work to locally oriented noncrossing trees. These are trees whose vertices are on the boundary of a circle such that the edges are line segments which do not cross inside the circle. Moreover, the edges are locally oriented. The author obtained the number of locally oriented noncrossing trees with a given number of vertices which are reachable from the root.

Nyariaro and Okoth [6], in 2020, further extended the work to enumerate plane trees, whose edges are locally oriented, according to path lengths, reachable sinks, leaves, first children, leftmost path, non-first children and non-leaves. They also obtained the formula for the number of trees in which an exact number of vertices is reachable from the root. An earlier work to take into account the case in which a given number of vertices are reachable from the root was considered by Seo and Shin [14]. Such trees are called *maximal increasing plane trees*.

Recently, Okoth and Nyariaro [10] studied reachability of vertices in t -ary trees whose edges are again locally oriented. They used path lengths, reachable sinks, leaves, leftmost path, first children, non-first children and non-leaves as the statistics of enumeration. The same method used for enumeration of plane trees was employed here as well. Enumeration of t -ary trees according to a given number of vertices which are reachable from the root was considered by Seo and Shin [13] in 2013. Again they called these trees *maximal increasing t -ary trees*.

In 2023, Nyariaro and Okoth [7] enumerated locally oriented 2-plane trees by reachability of its vertices. 2-plane trees are plane trees such that the vertices receive labels 1 and 2 so long as the sum of labels of the endpoints of an edge does not exceed 3. The statistics of enumeration are leftmost paths, leaf sinks, non-leaf sinks, first child and non-first child for 2-plane trees with roots of label 1 and label 2. The enumeration of locally oriented 2-plane trees by reachability was possible following a successful enumeration of 2-plane trees by Lumumba, Okoth and Kasyoki [5] according to the number of vertices of a given label in a 2-plane that reside at a certain level.

We need the following theorem in our enumeration:

Theorem 1.1 (Binomial Theorem). *For any integer $n \geq 0$,*

$$(x + y)^n = \sum_{k=0}^n \binom{n}{k} x^k y^{n-k}.$$

The following identity which is proved in basic combinatorics textbooks is helpful in getting our results:

$$\sum_{k=n}^m \binom{k}{n} = \binom{m+1}{n+1} \text{ (Hockey stick identity)}. \quad (1.1)$$

For the rest of this paper, we shall refer to complete labelled t -ary trees simply as trees. We enumerate these trees according to path lengths in Section 2. We obtain both explicit and asymptotic results. In Sections 3 and 4, we get equivalent results with the statistics of enumeration being sinks and leftmost paths respectively. We use first children and non-first children as statistics of enumeration in Sections 5 and 6 respectively. We conclude the paper in Section 7.

2. ENUMERATION BY PATH LENGTHS

In this section, using path length as the statistic of enumeration, we determine the average number of vertices in trees on $tn + 1$ vertices which are reachable from the root i such that there is a given number of edges on the path. We prove our first result.

Proposition 2.1. *The number of trees on $tn + 1$ vertices such that a given vertex v is reachable from the root i in ℓ steps is given by*

$$t^\ell (\ell t - \ell + t)(tn - \ell - 1)! \binom{v - i - 1}{\ell - 1} \binom{tn - \ell}{n - \ell - 1}. \quad (2.1)$$

Proof. Let $T(x)$ be the generating function for these trees where x marks the number of vertices. Symbolically, the generating function for these trees satisfies the equation: $T(x) = x + xT(x)^t$. By the decomposition of the trees in Figure 1, we have that the generating function for these trees in which there is a path of length $\ell \geq 0$ is given by:

$$(xT(x)^{t-1})^\ell x(T(x))^t = x^\ell T(x)^{\ell t - \ell} x T(x)^t = x^{\ell+1} T(x)^{\ell t - \ell + t}.$$

Applying Lagrange Inversion Formula [15], it follows that

$$\begin{aligned} [x^{tn+1}] x^{\ell+1} T(x)^{\ell t - \ell + t} &= [x^{tn-\ell}] T(x)^{\ell t - \ell + t} \\ &= \frac{\ell t - \ell + t}{tn - \ell} [s^{tn-\ell t - t}] (1 + s^t)^{tn-\ell}. \end{aligned}$$

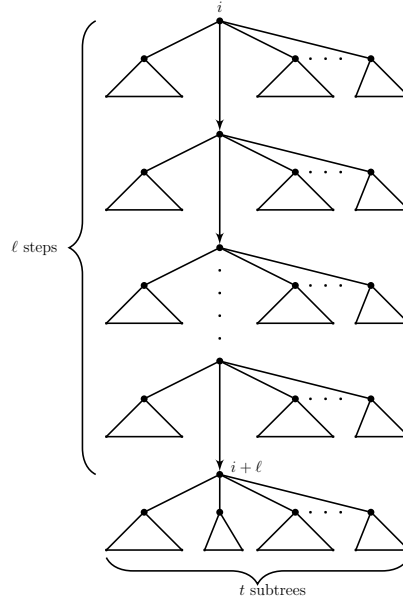


FIGURE 1. A tree with path length ℓ starting at vertex i to vertex $i + \ell$.

By Binomial Theorem (Theorem 1.1), we get

$$\begin{aligned} [x^{tn+1}]x^{\ell+1}T(x)^{\ell t - \ell + t} &= \frac{\ell t - \ell + t}{tn - \ell} [s^{tn - \ell t - t}] \sum_{i \geq 0} \binom{tn - \ell}{i} s^{ti} \\ &= \frac{\ell t - \ell + t}{tn - \ell} \binom{tn - \ell}{n - \ell - 1}. \end{aligned}$$

This formula gives the number of trees with path length $\ell \geq 0$. If the path starts at vertex i and ends at vertex v , then there are $\binom{v - i - 1}{\ell - 1}$ different labellings of the vertices on the path. Since the $\ell + 1$ vertices on the path have been labelled, we only need to obtain the different labellings of the other vertices in the tree. The vertices to be labelled are $(tn + 1) - (\ell + 1) = tn - \ell$. The labellings can be done in $(tn - \ell)!$ different ways. Moreover, an edge on a path can be any of the t edges which connect a vertex to its children. Therefore, there are t positions for each edge on the path. In total there are t^ℓ choices for positions of the edges in the path of length ℓ . So, we have

$$t^\ell (tn - \ell)! \binom{v - i - 1}{\ell - 1} \frac{\ell t - \ell + t}{tn - \ell} \binom{tn - \ell}{n - \ell - 1}$$

trees on $tn + 1$ vertices such that vertex v is reachable from vertex i in ℓ steps. This completes the proof. \square

Corollary 2.2. *The total number of labelled trees on $tn + 1$ vertices such that vertex v is reachable from the root in ℓ steps is given by:*

$$t^\ell(\ell t - \ell + t)(tn - \ell - 1)! \binom{v-1}{\ell} \binom{tn - \ell}{n - \ell - 1}. \quad (2.2)$$

Proof. To get the required formula, we sum over i in Equation (2.1), i.e.,

$$t^\ell(\ell t - \ell + t)(tn - \ell - 1)! \binom{tn - \ell}{n - \ell - 1} \sum_{i=1}^{v-\ell} \binom{v-i-1}{\ell-1}, \quad (2.3)$$

is the desired result which we now simplify. Let $k = v - i - 1$. When $i = 1$, then $k = v - 2$, and when $i = v - \ell$, then $k = \ell - 1$. So we rewrite the sum as,

$$\sum_{i=1}^{v-\ell} \binom{v-i-1}{\ell-1} = \sum_{k=\ell-1}^{v-2} \binom{k}{\ell-1}$$

Applying Hockey Stick Identity (1.1), we get

$$\sum_{i=1}^{v-\ell} \binom{v-i-1}{\ell-1} = \binom{v-1}{\ell}. \quad (2.4)$$

Substituting Equation (2.4) in Equation (2.3), we obtain the required formula. \square

Corollary 2.3. *The number of trees on $tn + 1$ vertices in which there is a path of length ℓ from vertex i is given by*

$$t^\ell(\ell t - \ell + t)(tn - \ell - 1)! \binom{tn - \ell}{n - \ell - 1} \binom{tn - i + 1}{\ell}. \quad (2.5)$$

Proof. We sum over v in Equation (2.1) to get:

$$\begin{aligned} t^\ell(\ell t - \ell + t)(tn - \ell - 1)! \binom{tn - \ell}{n - \ell - 1} \sum_{v=\ell+i}^{tn+1} \binom{v-i-1}{\ell-1} \\ = t^\ell(\ell t - \ell + t)(tn - \ell - 1)! \binom{tn - \ell}{n - \ell - 1} \sum_{k=\ell-1}^{tn-i} \binom{k}{\ell-1} \\ = t^\ell(\ell t - \ell + t)(tn - \ell - 1)! \binom{tn - \ell}{n - \ell - 1} \binom{tn - i + 1}{\ell}. \end{aligned}$$

The last equality follows by Hockey Stick Identity. This completes the proof. \square

Summing over all i in Equation (2.5), we get that there are

$$\begin{aligned} t^\ell(\ell t - \ell + t)(tn - \ell - 1)! \binom{tn - \ell}{n - \ell - 1} \sum_{i=1}^{tn-\ell+1} \binom{tn-i+1}{\ell} \\ = t^\ell(\ell t - \ell + t)(tn - \ell - 1)! \binom{tn - \ell}{n - \ell - 1} \binom{tn+1}{\ell+1} \end{aligned} \quad (2.6)$$

trees on $tn + 1$ vertices such that there is a path of length ℓ .

Setting $\ell = 0$, in Equation (2.6), we rediscover the formula for the number of trees on $tn + 1$ vertices, i.e.,

$$(tn)! \binom{tn+1}{n}. \quad (2.7)$$

Corollary 2.4. *On average, the number of trees on $tn + 1$ vertices in which there are oriented paths of length ℓ from the root tends to $\frac{t\ell - \ell + t}{t(\ell + 1)!}$ as n tends to infinity.*

Proof. We obtain the formula by dividing Equation (2.6) by Equation (2.7) and tending n to infinity.

We have,

$$\begin{aligned} & \frac{t^\ell(t\ell - \ell + t)(tn - \ell - 1)! \binom{tn-\ell}{n-\ell-1} \binom{tn+1}{\ell+1}}{(tn)! \binom{tn+1}{n}} \\ &= \frac{t^\ell(t\ell - \ell + t)(tn - \ell - 1)! \frac{(tn-\ell)!}{(n-\ell-1)!(tn-n+1)!} \cdot \frac{(tn+1)!}{(\ell+1)!(tn-\ell)!}}{(tn)! \frac{(tn+1)!}{n!(tn-n+1)!}} \\ &= \frac{t^\ell(t\ell - \ell + t)(tn - \ell - 1)!(tn - \ell)!n!(tn - n + 1)!(tn + 1)!}{(tn)!(tn + 1)!(\ell + 1)!(tn - \ell)!(tn - n + 1)!(n - \ell - 1)!} \\ &= \frac{t^\ell(t\ell - \ell + t)}{(\ell + 1)!} \cdot \frac{n!(tn - \ell - 1)!}{(tn)!(n - \ell - 1)!} \\ &= \frac{t^\ell(t\ell - \ell + t)}{(\ell + 1)!} \cdot \frac{n(n-1)(n-2)\cdots(n-\ell)}{tn(tn-1)(tn-2)\cdots(tn-\ell+1)(tn-\ell)}. \end{aligned}$$

Let

$$X = \frac{n(n-1)(n-2)\cdots(n-\ell)}{tn(tn-1)(tn-2)\cdots(tn-\ell+1)(tn-\ell)}.$$

Now,

$$\begin{aligned} \lim_{n \rightarrow \infty} X &= \lim_{n \rightarrow \infty} \frac{n(n-1)(n-2)\cdots(n-\ell)}{tn(tn-1)(tn-2)\cdots(tn-\ell+1)(tn-\ell)} = \lim_{n \rightarrow \infty} \frac{n^{\ell+1} + \cdots}{t^{\ell+1}n^{\ell+1} + \cdots} \\ &= \frac{1}{t^{\ell+1}} \lim_{n \rightarrow \infty} \frac{1 + \cdots}{1 + \cdots} = \frac{1}{t^{\ell+1}}. \end{aligned}$$

As $n \rightarrow \infty$, the average number of trees on $tn + 1$ vertices in which there are oriented paths of length ℓ from the root tends to

$$\frac{t^\ell(t\ell - \ell + t)}{(\ell + 1)!} \cdot \lim_{n \rightarrow \infty} X = \frac{t^\ell(t\ell - \ell + t)}{(\ell + 1)!} \cdot \frac{1}{t^{\ell+1}} = \frac{t\ell - \ell + t}{t(\ell + 1)!}.$$

□

3. ENUMERATION BY SINKS

In this section, we determine the number of vertices in labelled trees on $tn + 1$ vertices which are reachable from a given vertex such that there is a given number of edges on a path leading to a sink which is either a non-leaf sink or a leaf sink.

Proposition 3.1. *The number of trees on $tn + 1$ vertices such that there is a path of length ℓ starting at the root i and ending at a non-leaf sink v is given by:*

$$t^\ell (tn - \ell - t)! \binom{v - \ell - 1}{t} \frac{\ell t - \ell + t}{tn - \ell} \binom{tn - \ell}{n - \ell - 1} \binom{v - i - 1}{\ell - 1}. \quad (3.1)$$

Proof. Let $T(x)$ be the generating function for trees where x marks the number of vertices. Then $T(x) = x + xT(x)^t$. By the decomposition of the trees as in Figure 2 where vertex $i + \ell$ is a non-leaf sink of degree t , we have that the generating

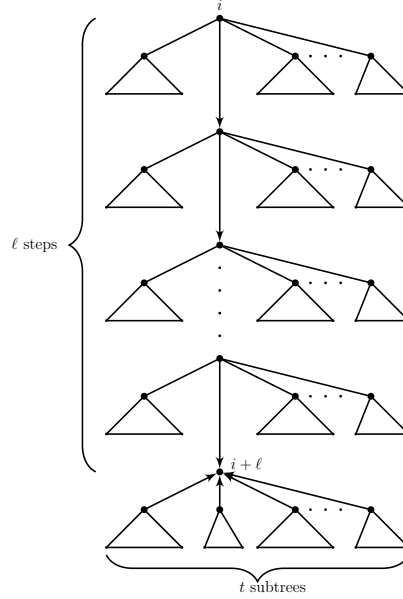


FIGURE 2. Unlabelled tree with path length ℓ with final vertex a non-leaf sink.

function for trees in which there is a path of length ℓ from vertex i is given by:

$$(xT(x)^{t-1})^\ell xT(x)^t = x^\ell T(x)^{\ell t - \ell} xT(x)^t = x^{\ell+1} T(x)^{\ell t - \ell + t}.$$

Applying Lagrange Inversion Formula [15], we have

$$\begin{aligned} [x^{tn+1}] x^{\ell+1} T(x)^{\ell t - \ell + t} &= [x^{tn-\ell}] T(x)^{\ell t - \ell + t} \\ &= \frac{\ell t - \ell + t}{tn - \ell} [s^{tn-\ell t - t}] (1 + s^t)^{tn-\ell}. \end{aligned}$$

By Binomial Theorem, we get

$$\begin{aligned} [x^{tn+1}] x^{\ell+1} T(x)^{\ell t - \ell + t} &= \frac{\ell t - \ell + t}{tn - \ell} [s^{tn-\ell t - t}] \sum_{i \geq 0} \binom{tn - \ell}{i} s^{ti} \\ &= \frac{\ell t - \ell + t}{tn - \ell} \binom{tn - \ell}{n - \ell - 1}. \end{aligned}$$

This is the number of trees with path length ℓ starting at vertex i to a non-leaf sink $i + \ell$. Instead of the path terminating at vertex $i + \ell$, let the path start at

i and end at non-leaf sink v . There are $\binom{v-i-1}{\ell-1}$ different labellings of the vertices on the path. Since the terminating vertex v is a sink of degree t , then the labels of the children of v must be less than v . Therefore, there are $\binom{v-\ell-1}{t}$ choices for labels of the children of v . Note that the $\ell+1$ vertices on the path of length ℓ and t children of v have been labelled. Overall, there are still a total of $(tn-\ell-t)$ vertices to be labelled. There are $(tn-\ell-t)!$ choices for labelling them. Moreover, an edge on a path, can be any of the t edges which connect a vertex to its children. Therefore, there are t positions for each edge on the path. In total there are t^ℓ choices for positions of the vertices in the path of length ℓ . Putting everything together, we have

$$t^\ell (tn-\ell-t)! \binom{v-\ell-1}{t} \frac{\ell t - \ell + t}{tn-\ell} \binom{tn-\ell}{n-\ell-1} \binom{v-i-1}{\ell-1}$$

trees on $tn+1$ vertices such that non-leaf sink v is reachable from vertex i in ℓ steps. \square

Corollary 3.2. *The total number of trees on $tn+1$ vertices in which a fixed root v is a non-leaf sink is given by:*

$$\frac{(tn-t)!}{n} \binom{v-1}{t} \binom{tn}{n-1}. \quad (3.2)$$

Proof. The result follows by setting $\ell=0$ and $i=v$ in Equation (3.1), i.e.,

$$(tn-t)! \binom{v-1}{t} \frac{t}{tn} \binom{tn}{n-1} \binom{-1}{-1}$$

with $\binom{-1}{-1} = 1$. \square

Corollary 3.3. *The total number of trees on $tn+1$ vertices in which the root is a non-leaf sink is given by:*

$$\frac{(tn-t)!}{n} \binom{tn+1}{t+1} \binom{tn}{n-1}. \quad (3.3)$$

Proof. From Equation (3.2), we get that the number of the required trees is

$$\sum_{v=t+1}^{tn+1} \frac{(tn-t)!}{n} \binom{v-1}{t} \binom{tn}{n-1} = \frac{(tn-t)!}{n} \binom{tn}{n-1} \sum_{v=t+1}^{tn+1} \binom{v-1}{t}. \quad (3.4)$$

Let $k=v-1$, so that

$$\sum_{v=t+1}^{tn+1} \binom{v-1}{t} = \sum_{k=t}^{tn} \binom{k}{t} = \binom{tn+1}{t+1}. \quad (3.5)$$

The last equality follows by Hockey Stick Identity.

Substituting Equation (3.5) in Equation (3.4), we obtain the required formula. \square

Corollary 3.4. *The total number of trees on $tn + 1$ vertices in which a child $v > i$ of a root i is a non-leaf sink is given by*

$$\frac{(2t^2 - t)(tn - t - 1)!}{tn - 1} \binom{tn - 1}{n - 2} \binom{v - 2}{t}. \quad (3.6)$$

Proof. We get the result by setting $\ell = 1$ in Equation (3.1). \square

Summing over all v in Equation (3.6), we find that there are

$$\begin{aligned} & \frac{(2t^2 - t)(tn - t - 1)!}{tn - 1} \binom{tn - 1}{n - 2} \sum_{v=t+2}^{tn+1} \binom{v - 2}{t} \\ &= \frac{(2t^2 - t)(tn - t - 1)!}{tn - 1} \binom{tn - 1}{n - 2} \sum_{k=t}^{tn-1} \binom{k}{t} \\ &= \frac{(2t^2 - t)(tn - t - 1)!}{tn - 1} \binom{tn - 1}{n - 2} \binom{tn}{t + 1} \end{aligned}$$

children of the root which are also non-leaf sinks in trees on $tn + 1$ vertices.

Corollary 3.5. *The total number of trees on $tn + 1$ vertices such that vertex v which is a non-leaf sink is reachable from the root in ℓ steps is given by*

$$t^\ell (tn - \ell - t)! \binom{v - \ell - 1}{t} \frac{\ell t - \ell + t}{tn - \ell} \binom{tn - \ell}{n - \ell - 1} \binom{v - 1}{\ell}. \quad (3.7)$$

Proof. To get the required formula, we sum over i in Equation (3.1), i.e.,

$$t^\ell (tn - \ell - t)! \binom{v - \ell - 1}{t} \frac{\ell t - \ell + t}{tn - \ell} \binom{tn - \ell}{n - \ell - 1} \sum_{i=1}^{v-\ell} \binom{v - i - 1}{\ell - 1}, \quad (3.8)$$

is the desired result which we can now simplify. Let $k = v - i - 1$. When $i = 1$, then $k = v - 2$ and when $i = v - \ell$ then $k = \ell - 1$. So, we have

$$\sum_{i=1}^{v-\ell} \binom{v - i - 1}{\ell - 1} = \sum_{k=\ell-1}^{v-2} \binom{k}{\ell - 1}$$

Applying Hockey Stick Identity, we get

$$\sum_{i=1}^{v-\ell} \binom{v - i - 1}{\ell - 1} = \binom{v - 1}{\ell} \quad (3.9)$$

Substituting Equation (3.9) in Equation (3.8), we get the equation in the statement of the corollary. \square

Corollary 3.6. *The total number of trees on $tn + 1$ vertices such that there is a path of length ℓ from the root to a non-leaf sink is given by*

$$t^\ell (tn - \ell - t)! \frac{\ell t - \ell + t}{tn - \ell} \binom{tn - \ell}{n - \ell - 1} \sum_{v=t+1}^{tn+1} \binom{v - \ell - 1}{t} \binom{v - 1}{\ell}.$$

Proof. To get the result, we sum over all v in Equation (3.7). \square

For the remainder of this section, our statistic of enumeration is leaf sink.

Proposition 3.7. *The number of trees on $tn + 1$ vertices in which a leaf sink v is reachable, at length $\ell \geq 0$, from the root i is given by*

$$t^\ell (tn - \ell)! \frac{\ell t - \ell}{tn - \ell} \binom{tn - \ell}{n - \ell} \binom{v - i - 1}{\ell - 1}. \quad (3.10)$$

Proof. Let $T(x)$ be the generating function for a tree where x marks the number of vertices. Then $T(x) = x + xT(x)^t$. These trees in which there is a path of length ℓ from the root i to a leaf sink $i + \ell$ is decomposed as shown in Figure 3.

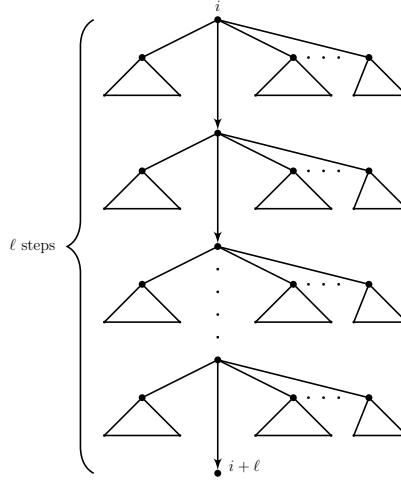


FIGURE 3. Unlabelled tree with path length ℓ with final vertex a leaf sink.

By the decomposition, we have that the generating function for the trees in which there is a path of length ℓ from a root i to a leaf sink $i + \ell$ is given by

$$(xT(x)^{t-1})^\ell x = x^\ell T(x)^{\ell t - \ell} x = x^{\ell+1} T(x)^{\ell t - \ell}. \quad (3.11)$$

Applying Lagrange Inversion Formula [15], it follows that

$$[x^{tn+1}] x^{\ell+1} T(x)^{\ell t - \ell} = [x^{tn-\ell}] T(x)^{\ell t - \ell} = \frac{\ell t - \ell}{tn - \ell} [s^{tn-\ell}] (1 + s^t)^{tn-\ell}. \quad (3.12)$$

By Binomial Theorem, we have

$$[x^{tn+1}] x^{\ell+1} T(x)^{\ell t - \ell} = \frac{\ell t - \ell}{tn - \ell} [s^{tn-\ell}] \sum_{i \geq 0} \binom{tn - \ell}{i} s^{ti} = \frac{\ell t - \ell}{tn - \ell} \binom{tn - \ell}{n - \ell}.$$

This formula gives the number of trees in which there is a path of length ℓ from the root. Let the path start at i and end at leaf sink v , then there are $\binom{v-i-1}{\ell-1}$ different labellings of the vertices on the path. Since the $\ell + 1$ vertices on the path have been labelled, there are still a total of $(tn - \ell)$ vertices to be labelled. This can be done in $(tn - \ell)!$ ways. An edge on a path can be any of the t edges which connects a vertex to its children. Thus, there are t positions for each edge on the path. In total, there are t^ℓ choices for positions of the vertices in the path of length ℓ . Collecting everything together, we get the desired formula. \square

Corollary 3.8. *The total number of trees on $tn + 1$ vertices in which vertex v is a leaf sink reachable from the root i in ℓ steps is given by*

$$t^\ell (tn - \ell - 1)! (\ell t - \ell) \binom{tn - \ell}{n - \ell} \binom{v - 1}{\ell}. \quad (3.13)$$

Proof. We obtain the required formula by summing over all i in Equation (3.10), i.e.,

$$t^\ell (tn - \ell)! \frac{\ell t - \ell}{tn - \ell} \binom{tn - \ell}{n - \ell} \sum_{i=\ell}^{v-\ell} \binom{v - i - 1}{\ell - 1}. \quad (3.14)$$

By Hockey Stick Identity, we get

$$\sum_{i=1}^{v-\ell} \binom{v - i - 1}{\ell - 1} = \binom{v - 1}{\ell}. \quad (3.15)$$

Substituting Equation (3.15) in Equation (3.14), we get Equation (3.10). This completes the proof. \square

Corollary 3.9. *The number of trees on $tn + 1$ such that there is a leaf sink at length ℓ which is reachable from the root i is given by*

$$t^\ell (tn - \ell - 1)! (\ell t - \ell) \binom{tn - \ell}{n - \ell} \binom{tn - i + 1}{\ell}. \quad (3.16)$$

Proof. We obtain the result by summing over v in Equation 3.10, i.e.,

$$t^\ell (tn - \ell - 1)! (\ell t - \ell) \binom{tn - \ell}{n - \ell} \sum_{v=\ell+i}^{tn+1} \binom{v - i - 1}{\ell - 1}. \quad (3.17)$$

Let $k = v - i - 1$. When $v = tn + 1$, $k = tn - i$ and when $v = \ell + i$, $k = \ell - 1$. We have,

$$\sum_{v=\ell+i}^{tn+1} \binom{v - i - 1}{\ell - 1} = \sum_{k=\ell-1}^{tn-i} \binom{k}{\ell - 1}.$$

By Hockey Stick Identity, we get

$$\sum_{k=\ell-1}^{tn-i} \binom{k}{\ell - 1} = \binom{tn - i + 1}{\ell}. \quad (3.18)$$

By Equations (3.17) and (3.18), we obtain the desired result. \square

Corollary 3.10. *The total number of leaf sinks, in trees on $tn + 1$ vertices is given by*

$$t^\ell (tn - \ell - 1)! (\ell t - \ell) \binom{tn - \ell}{n - \ell} \binom{tn + 1}{\ell + 1}. \quad (3.19)$$

Proof. We obtain the equation by summing over v in Equation 3.13, i.e.,

$$t^\ell (tn - \ell - 1)! (\ell t - \ell) \binom{tn - \ell}{n - \ell} \sum_{v=\ell+1}^{tn+1} \binom{v - 1}{\ell}. \quad (3.20)$$

Let $k = v - 1$. When $v = tn + 1$, $k = tn$ and when $v = \ell + 1$, $k = \ell$. So, we get

$$\sum_{v=\ell+1}^{tn+1} \binom{v-1}{\ell} = \sum_{k=\ell}^{tn} \binom{k}{\ell}.$$

Making use of Hockey Stick Identity, we get

$$\sum_{k=\ell}^{tn} \binom{k}{\ell} = \binom{tn+1}{\ell+1}. \quad (3.21)$$

By Equations (3.20) and (3.21), we obtain the required result. Alternatively, the corollary can be proved by summing over all i in Equation (3.16). \square

Setting $\ell = 0$, in Equation (3.19), we see that the root cannot be a leaf sink in any tree. Also, setting $\ell = 1$ in the same equation, we find the number of leaf sinks which are children of the root in trees on $tn + 1$ vertices.

Corollary 3.11. *On average, the number of leaf sinks which are reachable at length ℓ from the root in trees on $tn + 1$ vertices tends to $\frac{\ell(t-1)^2}{t(\ell+1)!}$ as n tends to infinity.*

Proof. We obtain the formula by dividing Equation (3.19) by Equation (2.7) and tending n to infinity. \square

4. ENUMERATION BY LEFTMOST PATHS

In this section, we enumerate trees according to the length of leftmost paths.

Proposition 4.1. *The number of trees on $tn + 1$ vertices such that there is the leftmost path of length ℓ starting at root i and ending at a vertex v is given by:*

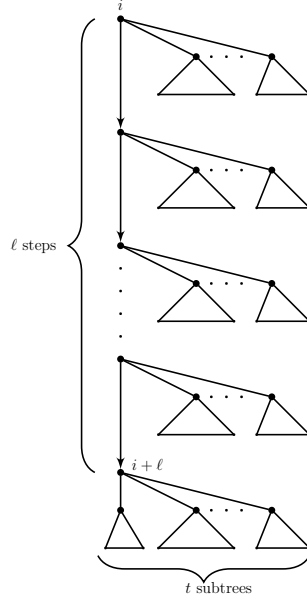
$$(tn - \ell - 1)!(\ell t - \ell + t) \binom{tn - \ell}{n - \ell - 1} \binom{v - i - 1}{\ell - 1}. \quad (4.1)$$

Proof. Let $T(x)$ be the generating function for trees where x marks the number of vertices. By the decomposition of the trees, as shown in Figure 4, the generating function is given by

$$(xT(x)^{t-1})^\ell xT(x)^t = x^{\ell+1}T(x)^{t\ell-\ell+t}. \quad (4.2)$$

By Lagrange Inversion Formula [15], we have

$$[x^{tn+1}]x^{\ell+1}T(x)^{t\ell-\ell+t} = [x^{tn-\ell}]T(x)^{t\ell-\ell+t} = \frac{t\ell + t - \ell}{tn - \ell} [s^{tn-t\ell-t}](1 + s^t)^{tn-\ell}. \quad (4.3)$$

FIGURE 4. Unlabelled tree with a leftmost path of length ℓ .

By Binomial Theorem, we get

$$\begin{aligned}
 [x^{tn+1}]x^{\ell+1}T(x)^{\ell t - \ell + t} &= \frac{\ell t - \ell + t}{tn - \ell} [s^{tn - \ell t - t}] \sum_{i \geq 0} \binom{tn - \ell}{i} s^{ti} \\
 &= \frac{\ell t - \ell + t}{tn - \ell} [s^{t(n - \ell - 1)}] \sum_{i \geq 0} \binom{tn - \ell}{i} s^{ti} \\
 &= \frac{\ell t - \ell + t}{tn - \ell} \binom{tn - \ell}{n - \ell - 1}. \tag{4.4}
 \end{aligned}$$

Equation (4.1) gives the number of trees on $tn + 1$ vertices such that there is a leftmost path of length ℓ from vertex i to terminal vertex $i + \ell$. Let the terminal vertex be v , then there are $\binom{v - i - 1}{\ell - 1}$ different labellings of the vertices on the path. The other vertices, the ones not on the path, are labelled in $((tn + 1) - (\ell + 1))! = (tn - \ell)!$ ways. We thus have the trees under consideration are

$$(tn - \ell)! \frac{\ell t - \ell + t}{tn - \ell} \binom{v - i - 1}{\ell - 1} \binom{tn - \ell}{n - \ell - 1}$$

in number. □

Corollary 4.2. *The total number of trees on $tn + 1$ vertices which there is a leftmost path of length ℓ from the root i is given by*

$$(tn - \ell - 1)! (\ell t - \ell + t) \binom{tn - \ell}{n - \ell - 1} \binom{tn - i + 1}{\ell}. \tag{4.5}$$

Proof. We obtain the desired formula by summing over v in Equation (4.1):

$$\begin{aligned}
& (tn - \ell - 1)!(\ell t - \ell + t) \binom{tn - \ell}{n - \ell - 1} \sum_{v=\ell+i}^{tn+1} \binom{v-i-1}{\ell-1} \\
&= (tn - \ell - 1)!(\ell t - \ell + t) \binom{tn - \ell}{n - \ell - 1} \sum_{k=\ell-1}^{tn-i} \binom{k}{\ell-1} \\
&= (tn - \ell - 1)!(\ell t - \ell + t) \binom{tn - \ell}{n - \ell - 1} \binom{tn-i+1}{\ell}.
\end{aligned}$$

□

Corollary 4.3. *There are*

$$(tn - \ell - 1)!(\ell t - \ell + t) \binom{tn - \ell}{n - \ell - 1} \binom{v-1}{\ell} \quad (4.6)$$

trees on $tn + 1$ vertices in which there are leftmost paths of length ℓ terminating at vertex v .

Proof. We sum over all i in Equation (4.1) to obtain the result:

$$\begin{aligned}
& (tn - \ell - 1)!(\ell t - \ell + t) \binom{tn - \ell}{n - \ell - 1} \sum_{i=1}^{v-\ell} \binom{v-i-1}{\ell-1} \\
&= (tn - \ell - 1)!(\ell t - \ell + t) \binom{tn - \ell}{n - \ell - 1} \sum_{k=\ell-1}^{v-2} \binom{k}{\ell-1} \\
&= (tn - \ell - 1)!(\ell t - \ell + t) \binom{tn - \ell}{n - \ell - 1} \binom{v-1}{\ell}.
\end{aligned}$$

□

Corollary 4.4. *The total number of trees on $tn + 1$ vertices such that there is a leftmost path of length ℓ is given by*

$$(tn - \ell - 1)!(\ell t - \ell + t) \binom{tn - \ell}{n - \ell - 1} \binom{tn+1}{\ell+1}. \quad (4.7)$$

Proof. We obtain the result by summing over all i in Equation (4.5) or over all v in Equation (4.6):

$$\begin{aligned}
& (tn - \ell - 1)!(\ell t - \ell + t) \binom{tn - \ell}{n - \ell - 1} \sum_{i=1}^{tn-\ell+1} \binom{tn-i+1}{\ell} \\
&= (tn - \ell - 1)!(\ell t - \ell + t) \binom{tn - \ell}{n - \ell - 1} \sum_{k=\ell}^{tn} \binom{k}{\ell} \\
&= (tn - \ell - 1)!(\ell t - \ell + t) \binom{tn - \ell}{n - \ell - 1} \binom{tn+1}{\ell+1}.
\end{aligned}$$

□

Setting $\ell = 0$ in Equation (4.7), we rediscover the formula for the number of trees on $tn + 1$ vertices, stated earlier as Equation (2.7).

Corollary 4.5. *On average, the number of non-leaf vertices at length k on left-most paths which are reachable from the roots of trees on $tn + 1$ vertices tends to $\frac{t\ell - \ell + t}{t^{\ell+1}(\ell + 1)!}$ as n tends to infinity.*

Proof. We obtain the formula by dividing Equation (4.7) by Equation (2.7) and tending n to infinity. \square

5. ENUMERATION BY FIRST CHILDREN

In this section, we enumerate trees according to level of first children.

Proposition 5.1. *The number of trees on $tn + 1$ vertices in which there is a path of length ℓ starting at the root i and terminating at a first child v is given by:*

$$t^{\ell-1}(tn - \ell - 1)!(\ell t - \ell + t) \binom{v - i - 1}{\ell - 1} \binom{tn - \ell}{n - \ell - 1}. \tag{5.1}$$

Proof. Let $T(x)$ be the generating function for trees. Then $T(x) = x + xT(x)^t$. By the decomposition of trees as shown in Figure 5,

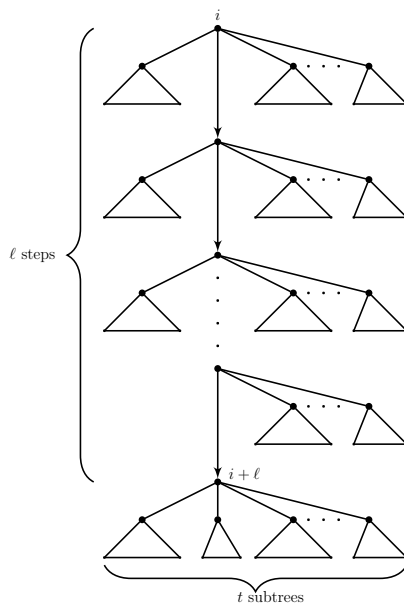


FIGURE 5. A tree with path length ℓ starting at vertex i to first child $i + \ell$.

the generating function for trees in which there is a path of length $\ell > 0$ is given by:

$$(xT(x)^{t-1})^{\ell-1}xT(x)^{t-1}xT(x)^t = x^{\ell+1}T(x)^{\ell t - \ell + t}.$$

Applying Lagrange Inversion Formula [15], it follows that

$$\begin{aligned} [x^{tn+1}]x^{\ell+1}T(x)^{\ell t-\ell+t} &= [x^{tn-\ell}]T(x)^{\ell t-\ell+t} \\ &= \frac{\ell t - \ell + t}{tn - \ell} [s^{tn-\ell t-t}] (1 + s^t)^{tn-\ell}. \end{aligned}$$

By Binomial Theorem, we get

$$\begin{aligned} [x^{tn+1}]x^{\ell+1}T(x)^{\ell t-\ell+t} &= \frac{\ell t - \ell + t}{tn - \ell} [s^{tn-\ell t-t}] \sum_{i \geq 0} \binom{tn - \ell}{i} s^{ti} \\ &= \frac{\ell t - \ell + t}{tn - \ell} \binom{tn - \ell}{n - \ell - 1}. \end{aligned} \quad (5.2)$$

Equation (5.2) gives the number of trees on $tn + 1$ vertices such that there is a path of length ℓ from vertex i to a first child $i + \ell$. Let the terminal first child be v , then there are $\binom{v - i - 1}{\ell - 1}$ different labellings of the vertices on the path. The other vertices which are not on that path can be labelled in $(tn - \ell)!$ ways. An edge on a path can be any of the t edges which connect a vertex to its children. Therefore, there are t positions for each edge on the path, except the final edge whose endpoint is the first child. In total there are $t^{\ell-1}$ choices for positions of the edges on the path of length ℓ , (the final edge has one choice for position). Thus there are

$$t^{\ell-1}(tn - \ell)! \frac{\ell t - \ell + t}{tn - \ell} \binom{v - i - 1}{\ell - 1} \binom{tn - \ell}{n - \ell - 1}$$

trees of order $tn + 1$ such that there is a path of length ℓ starting at a vertex i to a first child v . \square

Corollary 5.2. *The total number of trees on $tn + 1$ vertices in which there is a first child at length ℓ from the root i is given by*

$$t^{\ell-1}(tn - \ell - 1)!(\ell t - \ell + t) \binom{tn - \ell}{n - \ell - 1} \binom{tn - i + 1}{\ell}. \quad (5.3)$$

Proof. We obtain the desired formula by summing over v in Equation (5.1):

$$\begin{aligned} &t^{\ell-1}(tn - \ell - 1)!(\ell t - \ell + t) \binom{tn - \ell}{n - \ell - 1} \sum_{v=\ell+i}^{tn+1} \binom{v - i - 1}{\ell - 1} \\ &= t^{\ell-1}(tn - \ell - 1)!(\ell t - \ell + t) \binom{tn - \ell}{n - \ell - 1} \sum_{k=\ell-1}^{tn-i} \binom{k}{\ell - 1} \\ &= t^{\ell-1}(tn - \ell - 1)!(\ell t - \ell + t) \binom{tn - \ell}{n - \ell - 1} \binom{tn - i + 1}{\ell}. \end{aligned}$$

\square

Corollary 5.3. *There are*

$$t^{\ell-1}(tn - \ell - 1)!(\ell t - \ell + t) \binom{tn - \ell}{n - \ell - 1} \binom{v - 1}{\ell} \quad (5.4)$$

trees of order $tn + 1$ in which there are paths of length ℓ from the root to a first child v .

Proof. We sum over all i in Equation (5.1) to obtain the result:

$$\begin{aligned} & t^{\ell-1}(tn - \ell - 1)!(\ell t - \ell + t) \binom{tn - \ell}{n - \ell - 1} \sum_{i=1}^{v-\ell} \binom{v - i - 1}{\ell - 1} \\ &= t^{\ell-1}(tn - \ell - 1)!(\ell t - \ell + t) \binom{tn - \ell}{n - \ell - 1} \binom{v - 1}{\ell}. \end{aligned}$$

□

Corollary 5.4. *The total number of trees on $tn + 1$ vertices such that there is a path of length ℓ from the root terminating at a first child is given by*

$$t^{\ell-1}(tn - \ell - 1)!(\ell t - \ell + t) \binom{tn - \ell}{n - \ell - 1} \binom{tn + 1}{\ell + 1}. \quad (5.5)$$

Proof. We obtain the result by summing over all i in Equation (5.3) or over all v in Equation (5.4):

$$\begin{aligned} & t^{\ell-1}(tn - \ell - 1)!(\ell t - \ell + t) \binom{tn - \ell}{n - \ell - 1} \sum_{i=1}^{tn-\ell+1} \binom{tn - i + 1}{\ell} \\ &= t^{\ell-1}(tn - \ell - 1)!(\ell t - \ell + t) \binom{tn - \ell}{n - \ell - 1} \binom{tn + 1}{\ell + 1}. \end{aligned}$$

□

Corollary 5.5. *On average, the number of first children, at length ℓ , which are reachable from the roots of trees on $tn + 1$ vertices tends to $\frac{t\ell - \ell + t}{t^2(\ell + 1)!}$ as n tends to infinity.*

Proof. We obtain the formula by dividing Equation (5.5) by Equation (2.7) and tending n to infinity. □

6. ENUMERATION BY NON-FIRST CHILDREN

In this section, we enumerate trees according to non-first children which reside at a given level from a specified vertex.

Proposition 6.1. *The number of trees on $tn + 1$ vertices in which a non-first child v is reachable from the root i in ℓ steps is given by:*

$$(t - 1)t^{\ell-1}(tn - \ell)! \frac{\ell t - \ell + 2t - 1}{tn - \ell - 1} \binom{v - i - 1}{\ell - 1} \binom{tn - \ell - 1}{n - \ell - 2}. \quad (6.1)$$

Proof. We consider a tree of order $tn + 1$ with root i such that there is a path of length ℓ from the root to a non-first child $i + \ell$. The path decomposes the tree into left and right sub trees until step ℓ . Since vertex $i + \ell$ is a non-first child, there has to be an elder sibling of $i + \ell$. The vertex v can have a sub-tree attached to it or it can be empty. Let $T(x)$ be the generating function for trees. Again

$T(x) = x + xT(x)^t$. Here, x marks the number of vertices. Then we have that the generating function for trees in which there is a path of length $\ell > 0$ terminating at a non-first child is given by:

$$(xT(x)^{t-1})^{\ell-1}xT(x)^{t-2}xT(x)^t xT(x)^t = x^{\ell+2}T(x)^{\ell t - \ell + 2t - 1}.$$

This decomposition is given in Figure 6.

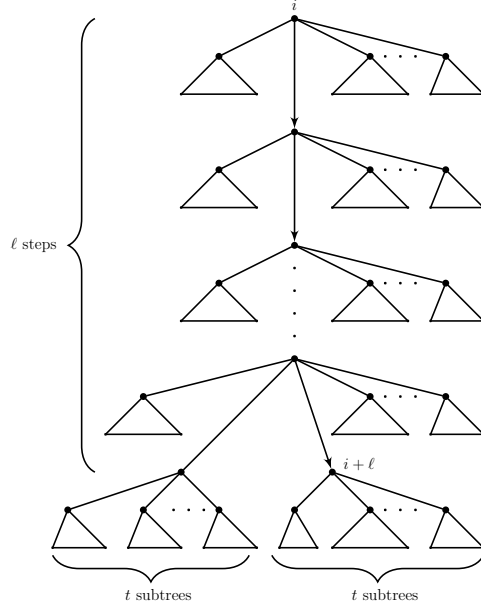


FIGURE 6. A tree with path length ℓ starting at vertex i to non-first child of label $i + \ell$.

By Lagrange Inversion Formula [15] and binomial theorem, it follows that

$$\begin{aligned} [x^{tn+1}]x^{\ell+2}T(x)^{\ell t - \ell + 2t - 1} &= [x^{tn - \ell - 1}]T(x)^{\ell t - \ell + 2t - 1} \\ &= \frac{\ell t - \ell + 2t - 1}{tn - \ell - 1} [s^{tn - \ell - 2t}] \sum_{i \geq 0} \binom{tn - \ell - 1}{i} s^{ti} \\ &= \frac{\ell t - \ell + 2t - 1}{tn - \ell - 1} \binom{tn - \ell - 1}{n - \ell - 2}. \end{aligned}$$

There are $\binom{v - i - 1}{\ell - 1}$ choices for these labels on the path from i to non-first child v . Note that when the $\ell + 1$ vertices on the path of length ℓ have been relabelled, we still have $tn - \ell$ vertices to label. This can be done in $(tn - \ell)!$ ways. An edge on a path can be any of the t edges which connect a vertex to its children. Therefore, there are t positions for each edge on the path, except the final edge whose endpoint is the non-first child. In total there are $(t - 1)t^{\ell-1}$ choices for positions of the edges in the path of length ℓ , (the final edge has $(t - 1)$ choices for position). Putting everything together, we obtain the desired formula. \square

Corollary 6.2. *The total number of trees on $tn + 1$ vertices in which there is a non-first child at length ℓ from the root i is given by*

$$(t-1)t^{\ell-1}(tn-\ell)! \frac{\ell t - \ell + 2t - 1}{tn - \ell - 1} \binom{tn - \ell - 1}{n - \ell - 2} \binom{tn - i + 1}{\ell}. \quad (6.2)$$

Proof. We obtain the desired formula by summing over v in Equation (6.1):

$$\begin{aligned} & (t-1)t^{\ell-1}(tn-\ell)! \frac{\ell t - \ell + 2t - 1}{tn - \ell - 1} \binom{tn - \ell - 1}{n - \ell - 2} \sum_{v=\ell+i}^{tn+1} \binom{v-i-1}{\ell-1} \\ &= (t-1)t^{\ell-1}(tn-\ell)! \frac{\ell t - \ell + 2t - 1}{tn - \ell - 1} \binom{tn - \ell - 1}{n - \ell - 2} \sum_{k=\ell-1}^{tn-i} \binom{k}{\ell-1} \\ &= (t-1)t^{\ell-1}(tn-\ell)! \frac{\ell t - \ell + 2t - 1}{tn - \ell - 1} \binom{tn - \ell - 1}{n - \ell - 2} \binom{tn - i + 1}{\ell}. \end{aligned}$$

□

Corollary 6.3. *There are*

$$(t-1)t^{\ell-1}(tn-\ell)! \frac{\ell t - \ell + 2t - 1}{tn - \ell - 1} \binom{tn - \ell - 1}{n - \ell - 2} \binom{v-1}{\ell} \quad (6.3)$$

trees of order $tn+1$ in which there are paths of length ℓ from the root that terminate at a non-first child v .

Proof. We sum over all i in Equation (6.1) to obtain the result:

$$\begin{aligned} & (t-1)t^{\ell-1}(tn-\ell)! \frac{\ell t - \ell + 2t - 1}{tn - \ell - 1} \binom{tn - \ell - 1}{n - \ell - 2} \sum_{i=1}^{v-\ell} \binom{v-i-1}{\ell-1} \\ &= (t-1)t^{\ell-1}(tn-\ell)! \frac{\ell t - \ell + 2t - 1}{tn - \ell - 1} \binom{tn - \ell - 1}{n - \ell - 2} \sum_{k=\ell-1}^{v-2} \binom{k}{\ell-1} \\ &= (t-1)t^{\ell-1}(tn-\ell)! \frac{\ell t - \ell + 2t - 1}{tn - \ell - 1} \binom{tn - \ell - 1}{n - \ell - 2} \binom{v-1}{\ell}. \end{aligned}$$

□

Corollary 6.4. *The total number of trees on $tn + 1$ vertices such that there is a path of length ℓ from the root terminating at a non-first child is given by*

$$(t-1)t^{\ell-1}(tn-\ell)! \frac{\ell t - \ell + 2t - 1}{tn - \ell - 1} \binom{tn - \ell - 1}{n - \ell - 2} \binom{tn + 1}{\ell + 1}. \quad (6.4)$$

Proof. We obtain the result by summing over all i in Equation (6.2) or over all v in Equation (6.3):

$$\begin{aligned} (t-1)t^{\ell-1}(tn-\ell)! \frac{\ell t - \ell + 2t - 1}{tn - \ell - 1} \binom{tn - \ell - 1}{n - \ell - 2} \sum_{i=1}^{tn-\ell+1} \binom{tn - i + 1}{\ell} \\ = (t-1)t^{\ell-1}(tn-\ell)! \frac{\ell t - \ell + 2t - 1}{tn - \ell - 1} \binom{tn - \ell - 1}{n - \ell - 2} \sum_{k=\ell}^{tn} \binom{k}{\ell} \\ = (t-1)t^{\ell-1}(tn-\ell)! \frac{\ell t - \ell + 2t - 1}{tn - \ell - 1} \binom{tn - \ell - 1}{n - \ell - 2} \binom{tn + 1}{\ell + 1}. \end{aligned}$$

□

Corollary 6.5. *The average number of non-first children which are reachable from the root in ℓ steps in a random tree is given by*

$$\frac{(t-1)(\ell t - \ell + 2t - 1)}{(\ell + 1)!t^3}.$$

Proof. We obtain the formula by dividing Equation (6.4) by Equation (2.7) and tending n to infinity. □

For the rest of this section, we enumerate trees by the number of non-first children which are reachable at a given length such that the non-first child is a leaf.

Proposition 6.6. *The number of trees of order $tn + 1$ such that there is a path of length ℓ starting at root i and ending at a non-first child v which is also a leaf is given by:*

$$(t-1)t^{\ell-1}(tn-\ell)! \frac{\ell t - \ell + t - 1}{tn - \ell - 1} \binom{v - i - 1}{\ell - 1} \binom{tn - \ell - 1}{n - \ell - 1}. \quad (6.5)$$

Proof. Let $T(x)$ be the generating function for trees. Then $T(x) = x + xT(x)^t$. Here, x marks vertices. Let's consider a path of length ℓ from vertex i to a non-first child $i + \ell$. Moreover, let vertex $i + \ell$ be a leaf. By the decomposition of the trees as in Figure 7, the generating function for these trees is given as

$$(xT(x)^{t-1})^{\ell-1} xT(x)^{t-2} xT(x)^t x = x^{\ell+2} T(x)^{\ell t + t - \ell - 1}.$$

By Lagrange Inversion Formula [15], we have

$$\begin{aligned} [x^{tn+1}] x^{\ell+2} T(x)^{\ell t - \ell + t - 1} &= [x^{tn-\ell-1}] T(x)^{\ell t - \ell + t - 1} \\ &= \frac{\ell t - \ell + t - 1}{tn - \ell - 1} [s^{tn-\ell-t}] (1 + s^t)^{tn-\ell-1}. \end{aligned}$$

Now, Binomial Theorem gives

$$\begin{aligned} [x^{tn-\ell-1}] T(x)^{\ell t - \ell + t - 1} &= \frac{\ell t - \ell + t - 1}{tn - \ell - 1} [s^{tn-\ell-t}] \sum_{i \geq 0} \binom{tn - \ell - 1}{i} s^{ti} \\ &= \frac{\ell t - \ell + t - 1}{tn - \ell - 1} \binom{tn - \ell - 1}{n - \ell - 1}. \end{aligned}$$

Corollary 6.8. *There are*

$$(t-1)t^{\ell-1}(tn-\ell)! \frac{\ell t - \ell + t - 1}{tn - \ell - 1} \binom{tn - \ell - 1}{n - \ell - 1} \binom{v-1}{\ell} \quad (6.7)$$

trees of order $tn+1$ in which there are paths of length ℓ from the root that terminate at a non-first child v which is also a leaf.

Proof. We sum over all i in Equation (6.5) to obtain the result:

$$\begin{aligned} & (t-1)t^{\ell-1}(tn-\ell)! \frac{\ell t - \ell + t - 1}{tn - \ell - 1} \binom{tn - \ell - 1}{n - \ell - 1} \sum_{i=1}^{v-\ell} \binom{v-i-1}{\ell-1} \\ &= (t-1)t^{\ell-1}(tn-\ell)! \frac{\ell t - \ell + t - 1}{tn - \ell - 1} \binom{tn - \ell - 1}{n - \ell - 1} \sum_{k=\ell-1}^{v-2} \binom{k}{\ell-1} \\ &= (t-1)t^{\ell-1}(tn-\ell)! \frac{\ell t - \ell + t - 1}{tn - \ell - 1} \binom{tn - \ell - 1}{n - \ell - 1} \binom{v-1}{\ell}. \end{aligned}$$

□

Corollary 6.9. *The total number of trees on $tn+1$ vertices such that there is a path of length ℓ from the root terminating at a non-first child which is also a leaf is given by*

$$(t-1)t^{\ell-1}(tn-\ell)! \frac{\ell t - \ell + t - 1}{tn - \ell - 1} \binom{tn - \ell - 1}{n - \ell - 1} \binom{tn+1}{\ell+1}. \quad (6.8)$$

Proof. We obtain the result by summing over all i in Equation (6.6) or over all v in Equation (6.7):

$$\begin{aligned} & (t-1)t^{\ell-1}(tn-\ell)! \frac{\ell t - \ell + t - 1}{tn - \ell - 1} \binom{tn - \ell - 1}{n - \ell - 1} \sum_{i=1}^{tn-\ell+1} \binom{tn-i+1}{\ell} \\ &= (t-1)t^{\ell-1}(tn-\ell)! \frac{\ell t - \ell + t - 1}{tn - \ell - 1} \binom{tn - \ell - 1}{n - \ell - 1} \sum_{k=\ell}^{tn} \binom{k}{\ell} \\ &= (t-1)t^{\ell-1}(tn-\ell)! \frac{\ell t - \ell + t - 1}{tn - \ell - 1} \binom{tn - \ell - 1}{n - \ell - 1} \binom{tn+1}{\ell+1}. \end{aligned}$$

□

Corollary 6.10. *On average, the number of non-first children (also non-leaves), which are reachable from the roots of trees on $tn+1$ vertices tends to $\frac{(t-1)^2(\ell t - \ell + t - 1)}{(\ell+1)!t^3}$ as n goes to infinity.*

Proof. We obtain the formula by dividing Equation (6.8) by Equation (2.7) and tending n to infinity. □

7. CONCLUSION

In this work, we have built on the previous work of Okoth, Okoth and Nyariaro and Seo and Shin to enumerate labelled complete t -ary trees according to path lengths, reachable sinks, leaves, first children, leftmost paths and non-first children. The work can be extended to obtain equivalent results for t -ary trees whose vertices are labelled with integers $1, 2, \dots, k$ such that if the rightmost child has label j and its parent label i then $i \leq j$. These trees were introduced and studied by Panholzer and Prodinger in [11].

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