ON THE ALGEBRAIC NUMERICAL RANGE OF THE BASIC ELEMENTARY OPERATOR AND NORM OF A GENERALIZED DERIVATION

BY

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DECLARATION

This thesis is my own work and has not been presented for a degree award in any other institution.

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DEDICATION

This piece of work is dedicated to my lovely children; Beryl, Shanice, Stephan and Daniella who have been a pillar of strength and great encouragement to me. I owe each of them more than I can practically express; God bless you abundantly.

ABSTRACT

An elementary operator is a bounded linear map defined on the set of bounded linear operators acting on an infinite dimensional complex Hilbert space H. Various forms of elementary operators have been studied in the past including generalized elementary operator, left and right operators, inner derivation, generalized derivation and basic elementary operators among others. Various aspects such as spectra, compactness, norm properties, numerical ranges among others have been used to study the properties of these operators in the recent past and good results have been obtained. Over time, the relationship between the various generalizations of numerical range have been investigated and it emerges that for the basic elementary operator, an exact description of the numerical range has not been exhaustively explored as the operator acts on various algebras. In this study, we have investigated both the algebraic numerical range of the basic elementary operator and the norm of a generalized derivation on the operator algebra $B(H)$. Specifically, by application of set inclusion approach and convexity properties of sets, it has been shown that the algebraic numerical range of the basic elementary operator is convex, contains the closure of its classical numerical range and that the algebraic numerical range of the basic elementary operator is equal to the closure of the classical numerical range of the implementing operators. Finally, we have shown that the norm of a generalized derivation is equal to the sum of the norms of implementing operators if the operators are finite rank. These results are of great importance to quantum physics, for solving force closure in robotic grasping and provision of basis of solution to optimization and duality problems.

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CHAPTER ONE

INTRODUCTION

1.1 Background of the study.

The study of elementary operators originated from the theory of matrix equations by Sylvester in 1884 [29]. He computed the eigenvalues of the matrix operators corresponding to the elementary operator T_{AB} on the set of an $n \times n$ matrix. The term elementary operator was coined by Lumer and Rosenblum [18] in a more general Banach algebra context in 1959. They computed the spectra of such operators and put a lot of emphasis on the application of these spectra to the systems of operator equations. Since then, many mathematicians in operator theory have been interested in the study of the properties of elementary operators such as their spectrum, compactness, norm and numerical ranges see [2], [6] and [7]. There are varied good results and expositions on different aspects of elementary operators, mainly their spectral and structural properties. However, there are still many open problems since these properties are connected directly to the structure of the underlying space that the operator is defined on. The norm of inner and a generalized derivation have been computed in various spaces.

On the other hand, the concept of a quadratic form associated with a matrix and its application in matrix theory is known. These ideas when extended naturally in finite and infinite dimensional spaces, lead to the concept of numerical range for matrices. Numerical range is also referred to as Hausdorff domain, range of values and most importantly, the field of values in matrix theory, when the operator is defined on a finite dimensional space.

In early studies of Hilbert spaces, researchers such as Hellinger, Toeplitz, Hilbert among others, concentrated on the quadratic forms. The sesquilinear form f_T associated to a bounded linear operator T on a Hilbert space H is given by $f_T(x, y) = \langle Tx, y\rangle$, for all $x, y \in H$ which corresponds to the quadratic form $f_T(x) = f_T(x, x) = \langle Tx, x\rangle$ for all $x \in H$. The range of restriction of the quadratic form to the unit sphere gives the formal definition of the numerical range implemented by the operator T as $W(T) = \{\langle Tx, x\rangle,\ x \in H\ \|x\| = 1\}$.

Topological properties of numerical range have been proved such as its non-emptiness, inclusion of the spectrum of an operator within the closure of the operator's numerical range, the numerical range of an operator lying in the closed disk of radius equal to the norm of the operator and most importantly, its convexity by Toeplitz and Hausdorff see [1], [11], [23], [30] and [31]. Many studies have given generalizations that are varied depending on whether the set being considered is finite or infinite dimensional.

The generalizations of the numerical range, both in finite and infinite dimension in Banach spaces or Banach algebras have been extensively explored by Bonsall and Duncan [6]. For more details see [11] and [15]. The classical numerical range for elementary operators in Banach spaces, Banach algebras and C*-algebras have been studied extensively too. Currently, motivated by theoretical study and applications, there are lots of dynamic research in matrix analysis and operator theory on the numerical range and its generalizations that have been done and documented. The areas of interest include; k-numerical range, c-numerical range, m-numerical range and their generalization in matrix analysis. In operator theory, the generalizations are mainly on the extension to Banach spaces using the Hahn-Banach theorem and the concept of the semi-inner product. The algebraic numerical range of the basic elementary operator is such a generalization that we have obtained.

1.2 Statement of the Problem.

Various generalizations of numerical ranges of various elementary operators in certain base fields have been studied in the past and good results obtained. However, there is very little results on the relationship between the numerical ranges of these elementary operators with that of their implementing operators. In this study, we use Barraa's results to investigate the relationship between the algebraic numerical range of the basic elementary operator with that of the classical numerical range of its implementing operators in the operator algebra $B(H)$.

The norm of a generalized derivation δ_{AB} on $B(H)$ has been fully determined by Stampfli using maximal numerical range. The norm restriction of this operator on finite rank operators has not been done. In this study, therefore we determine the norm of a generalized derivation δ_{AB} on $B(H)$ using finite rank operators. We have also continued to use Stampfli's maximal numerical range to determine this norm equality.

1.3 Objective of the study

1.3.1 General objective

The main objective of the study was to investigate the algebraic numerical range of the basic elementary operator and the norm of a generalized derivation on $B(H)$.

1.3.2 Specific objectives

Our study aimed at

(i) Proving that the algebraic numerical range of the basic elementary operator is convex and contains the closure of its classical numerical range on $B(H)$.

(ii) Determining the relationship between the algebraic numerical range of the basic elementary operator and the classical numerical range of the implementing operators on $B(H)$.

(iii) Determining the norm properties of a generalized derivation on $B(H)$.

1.4 Significance of the study

The geometrical properties of numerical range provide important information about the algebraic and analytic properties of an operator. Thus the geometric properties of the numerical range is an important tool for classifying types of operators such as unitary, normal and self adjoint. The theory of numerical range has been crucial in the study of some algebraic structures of operators mostly in the non-associative context. The numerical range of an operator T contains important information on the properties of the operator. The numerical range allows one to deduce many properties of an operator. For example, the numerical range is often used to locate the spectrum of an operator since the spectrum is known to be contained within the numerical range of the operator. Numerical range can also be used to obtain dilations with simple structure, lower and upper norm bounds among others.
Numerical range has been used in engineering as a rough estimate of eigenvalues of an operator. Approximation using commutators $TX-XS$ or $TX-XT$ are problems in quantum mechanics that researchers in Applied Mathematics and Physics have an interest in.

Since the numerical range and numerical radius are closely related by definition, the distance of the numerical range of the operator T to the origin and the numerical radius of T are useful in the study of approximation problems, stability, perturbation and convergence. More precisely, the numerical radius has been used as a reliable indicator for iterative methods and rate of convergence and therefore, it is important in stability theory of finite-differences approximations for hyperbolic initial-value problems.

On the other hand since by definition the norm of a vector is its length from the zero vector, the operator norm of any operator is equal to the spectral radius especially if the operator is normal. This gives the least upper bound on the magnitude of the largest eigenvalue.

We hope that our research findings will greatly contribute to the field of numerical range of elementary operators and norm of derivation and provide motivation for further research to pure mathematicians in these areas of study. The research findings will contribute to the theoretical knowledge needed by mathematicians and physicists.

1.5 Research methodology

1.5.1: Convexity

To prove that the algebraic numerical range of the basic elementary operator is convex, we have built up the convexity using a functional and showed that the functional is a state since it preserves convexity.

1.5.2: Equality of sets

To prove that the algebraic numerical range of the basic elementary operator is equal to the closure of the classical numerical range of its implementing operators, we have showed set inclusion in both directions since numerical range is a set.

1.5.3: Norm properties

The norm properties of elementary operators have been comprehensively studied by many researchers. From the already established norms, done by applying the definition of an operator norm and the known results of Stampfli, we have determined the norm of a generalized derivation by showing that inequality holds in both directions. Also, from the theory of Banach spaces, the Hahnn- Banach theorem allows us to algebraically construct functionals in the subspace which is extended in the whole space under consideration. Finite rank operators is a mathematical tool that has been used in the construction to determine the norm of a generalized derivation.

1.6 Organization of the study

Our study is divided into four chapters. In this first chapter, we give the overview of our study by discussing the background of our work, and the significance of the study. We have stated the problem and outlined the objectives to be achieved together with the methods used. Finally we give the definitions of basic concepts and necessary theoretical information which are relevant to the study. In the second chapter we give the literature review of the study. The main results are given in chapter three, where in the section 3.1, we have given results on the numerical range of the basic elementary operator and the section 3.2, is on the norm of a generalized derivation. We finally give our summary and recommendations for future research in chapter four. The references used in the study are given at the end.

1.7 Basic concepts

In this section we give definitions and the necessary theoretical background information from the theories of normed spaces, Banach spaces, Banach algebras and C*-algebras that are relevant in the study.

1.7.1: Algebra.

An **algebra** is a vector space $\mathcal{A}$ equipped with a bilinear map called multiplication such that $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$: $(a, b) \to ab$ which is associative;$(ab)c = a(bc)$ for all $a, b, c \in \mathcal{A}$.

An algebra $\mathcal{A}$ is **commutative** if its multiplication is, that is, $ab = ba$ for all $a, b \in \mathcal{A}$.

An algebra $\mathcal{A}$ is **unital** if there exist a unique $e \in \mathcal{A}$ such that $ea = ae = a$ for all $a \in \mathcal{A}$.

1.7.2: Banach Algebras

A norm $\|.\|$ defined on an algebra $\mathcal{A}$ is said to be submultiplicative if

$$\|ab\| \leq \|a\|\|b\|$$

for all $a, b \in \mathcal{A}$.

A normed algebra is an algebra with submultiplicative norm defined on it.

A Banach algebra is a complete normed algebra and a unital normed algebra is a normed algebra $\mathcal{A}$ with a unit element e.

A unital Banach algebra is a complete normed algebra with unit element e.

Generally, there are three kinds of Banach algebras depending on the way the multiplication is defined. Thus from [21] we have;

(a) **Operator algebras**

Here we consider algebras whose elements are operators on a Banach space. In this case the multiplication is the composition of operators.

• If $\mathcal{X}$ is a complex Banach space, then $B(\mathcal{X})$ the Banach space of all bounded linear operators on $\mathcal{X}$ with respect to the operator norm is a Banach algebra with the multiplication defined as composition of operators. The identity operator I is the unit element. The operator norm of $T \in B(\mathcal{X})$ is given by

$$\|T\| = sup\{\|Tx\| : \ x \in X, \ \|x\| \leq 1\}.$$

For $T, S \in B(\mathcal{X})$ we have that,

$\|TSx\| \leq \|T\|\|Sx\| \leq \|T\|\|S\|\|x\|$

$\Rightarrow \|TS\| \leq \|T\|\|S\|$.

$\|I\| = 1$ since $\|Ix\| = \|x\| \leq 1 \Rightarrow \|I\| \leq 1$ and

$\|x\| = \|Ix\| \leq \|I\|\|x\| \Rightarrow 1 \leq \|I\|$.

If $dim\mathcal{X} \geq 2$, then $B(\mathcal{X})$ is a non commutative Banach algebra.

If $\mathcal{X} = H$, a Hilbert space, then $B(H)$ is also a Banach algebra.

• Let $\mathcal{A} = M_n(\mathbb{C})$, $(n \geq 2)$, the set of all $n \times n$ matrices with matrix addition and matrix multiplication and with Frobenious norm defined by

$$\|A\|_F = (\sum_{i,j=1}^{n} |a_{ij}|^2)^{\frac{1}{2}}$$

is a non commutative unital Banach algebra.

• The set $\mathcal{K}(\mathcal{X}) = \{T \in B(\mathcal{X}) : \ T \ is \ compact\}$ is a closed sub-algebra of $B(\mathcal{X})$ and therefore is a Banach space. $\mathcal{K}(\mathcal{X})$ is unital if and only if $dim\mathcal{X} < \infty$ and it is an ideal in $B(\mathcal{X})$.

(b) **Function algebras**

Here we consider algebras of functions. In this case the multiplication is pointwise.

• Let $\mathcal{A} = \mathbb{C}$. Then with respect to the usual addition, multiplication of complex numbers and modulus, $\mathcal{A}$ is a commutative, unital Banach algebra.

• Let K be a compact Hausdorff space and $\mathcal{A} = C(K)$. Then with respect to the pointwise multiplication of functions, $\mathcal{A}$ is a commutative unital Banach algebra and the norm is given by

$$\|f\|_\infty = sup_{t \in K}|f(t)|.$$

• Let $S \neq 0$ and $B(S) = \{f : S \to \mathbb{C} : \ S \ is \ bounded\}$. For $f, g \in B(S)$ and $s \in S$ we define

$(f + g)(s) = f(s) + g(s)$

$(\alpha f)(s) = \alpha f(s)$ for all $f, g \in B(S)$ and $\alpha \in \mathbb{C}$

• Let Ω be a locally compact Hausdorff space and let

$\mathcal{A} = \mathcal{C}_b(\Omega) = \{f : \mathcal{C}(\Omega) \ : \ f \ is \ bounded\}$.

Then $\mathcal{A}$ is a commutative unital Banach algebra.

• Let $\mathcal{A} = \mathcal{C}_o(\Omega) = \{f \in C(\Omega) : \ f \ vanishes \ at \ \infty\}$. Here f vanishes at infinity if and only if for every $\varepsilon > 0$, there exist a compact subset K_ε of Ω such that $|f(t)| < \varepsilon$ for every $t \in K_\varepsilon^c$. $\mathcal{A}$ is a commutative Banach algebra and is unital if and only if Ω is compact.

• Let $X = [0,1]$. Then $C'[0,1] \subset C[0,1]$ is an algebra and $(C'[0,1], \|.\|_\infty)$ is not complete. If we define $\|f\| = \|f\|_\infty + \|f'\|\infty$, $f \in C'[0,1]$, then $(C'[0,1], \|.\|)$ is a Banach algebra.

• Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Considering
$\mathcal{A}(\mathbb{D}) = \{f \in C(\overline{\mathbb{D}}) : f|_{\mathbb{D}} \textit{ is analytic}\}$. Then $\mathcal{A}(\mathbb{D})$ is a closed subalgebra of $C(\mathbb{D})$ and so is a commutative unital Banach algebra called disk algebra.

• Let $H^\infty(\mathbb{D}) = \{f : \mathbb{D} \to \mathbb{C} : f \textit{ is bounded and analytic}\}$. This is a commutative unital Banach algebra with respect to the point wise addition, multiplication of functions and usual scalar multiplication of function and the supremum norm.

(c) **Group algebras**

These are algebras which consists of functions but the multiplication is the convolution product. These algebras do not always have a unit.

• Let $\mathcal{A} = L^1(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{C} : f \textit{ is measurable and} \int |f(t)|dt < \infty\}$. Multiplication is defined by

$$(f * g)(x) = \int f(x-t)g(t)d\mu(t)$$

for all $f, g \in L^1(\mathbb{R})$.

The norm is defined by

$$\|f\|_1 = \int |f(t)|dt,$$

and

$$\|f * g\|_1 \leq \|f\|_1 \|g\|_1$$

for all $f, g \in L^1(\mathbb{R})$.

Since $\mathcal{A}$ is complete, then $L^1(\mathbb{R})$ is a Banach algebra and the convolution is commutative.

• Let $w : \mathbb{R} \to [0, \infty)$ be such that $w(s+t) \leq w(s) + w(t)$ for all $s, t \in \mathbb{R}$. Let $L^1_w(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{C} : f \textit{ is measurable and} \int_{\mathbb{R}} |f(t)|w(t)dt < \infty\}$. For $f \in L^1_w$, the norm is defined as

$$\|f\| = \int_{\mathbb{R}} |f(t)|w(t)dt.$$

Then $L^1_w(\mathbb{R})$ is a Banach space and with the convolution it becomes a Banach algebra.

• Let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ and
$\mathcal{A} = L^1(\mathbb{T}) = \{f : \mathbb{T} \to \mathbb{C} : f \textit{ is measurable and } \int_{\mathbb{T}} |f(t)|\mu(dt) < \infty\}$. μ is the normalized Lebesgue measure on $\mathbb{T}$. The convolution is defined by

$$(f * g)(t) = \int_{\mathbb{T}} |f(t)|\mu(dt)$$

for all $f, g \in \mathcal{A}$.

Then $\mathcal{A}$ is a non unital commutative Banach algebra.

• Let $\mathcal{B} = \{f : \mathbb{R} \to \mathbb{C} : f \textit{ is measurable}, \ 2\pi\textit{-periodic and } \int_0^{2\pi} |f(t)|dt < \infty\}$.
The convolution is defined by

$$(f * g)(t) = \int_0^{2\pi} f(t - \tau)g(\tau)d\tau$$

for all $f, g \in \mathcal{B}$.

The norm is defined by

$$\|f\| = \frac{1}{2\pi} \int_0^{2\pi} |f(t)|dt.$$

$\mathcal{B}$ is a commutative, non unital Banach algebra.

Generally, if G is a locally compact topological group, then there is a Haar measure μ on G, that is, a regular Borel measure that is invariant under left translation. $L^1(G, \mu)$ is a Banach algebra with convolution

$$(f * g)(x) = \int_G f(t)g(t^{-1}x)d\mu(t)$$

for $f, g \in L^1 d\mu(x)$.

Here, $L^1(G) = \{f / f : G \to \mathbb{C}\}$.

The Haar measure is unique up to the positive constant and so we write,

$$\int_G f(x)dx = \int_G f(x)d\mu(x).$$

$L^1(G)$ is is commutative if and only if G is abelian.

1.7.3: C^*-algebra

Let $\mathcal{A}$ be a complex algebra. An involution on $\mathcal{A}$ is a conjugate linear map $* : \mathcal{A} \to \mathcal{A}$ that satisfies the following axioms;

(i) $(a^*)^* = a$ for all $a \in \mathcal{A}$,

(ii) $(ab)^* = b^*a^*$ for all $a, b \in \mathcal{A}$ and

(iii) $(\mu a + \lambda b)^* = \overline{\mu}a^* + \overline{\lambda}b^*$ for all $a, b \in \mathcal{A}$ and $\mu, \lambda \in \mathbb{C}$

A *-algebra or an involution algebra is the ordered pair $(\mathcal{A}, *)$.
An element $a \in \mathcal{A}$ is said to be self adjoint or hermitian if $a = a^*$.
In addition, a is said to be normal if $a^*a = aa^*$ and unitary if $a^*a = aa^* = e$
A Banach *-algebra is an involution algebra $\mathcal{A}$ endowed with a complete submultiplicative norm such that $\|a^*\| = \|a\|$ for all $a \in \mathcal{A}$.
$\mathcal{A}$ is called a unital Banach *- algebra if it has a unit element e such that $\|e\| = 1$.
A C*-norm on $\mathcal{A}$ is the norm for which $\|a^*a\| = \|a\|^2$ for all $a \in \mathcal{A}$.
A **C*-algebra** $\mathcal{A}$ is a Banach*-algebra which is complete in the C*-norm.
A C*-algebra $\mathcal{A}$ is said to be unital if it has a unit element $e \in \mathcal{A}$ such that that $ea = ae = a$ for all $a \in \mathcal{A}$.
On the other hand if a C*-algebra is non unital, it can always be unitized by adjoining a unit to it.
A subset S of a C*-algebra is called **C*-sub-algebra** if it is a C*-algebra with the inherited operations, involution and norm.
The following are some examples of C*-algebras;

(i) If $\mathcal{A} = \mathbb{C}$, the map $z \to \overline{z}$ (where $\overline{z}$ is the complex conjugate of z) is an involution with which $\mathcal{A}$ becomes a commutative involutive algebra and hence a C*-algebra.

(ii) If $\mathcal{X}$ is a compact Hausdorff topological space and $\mathcal{A} = C(\mathcal{X})$ the algebra of complex valued continuous functions on $\mathcal{X}$, then $\mathcal{A}$ is a commutative C*-algebra with involution $f^* = \overline{f}$ (Pointwise conjugation of function values). When $\mathcal{X}$ is a single point, this reduces to example (i) above.

(iii) Let H be a Hilbert space and $\mathcal{A} = B(H)$ the bounded linear endomorphism of H is a C*-algebra with the usual adjoint defined by

$$\langle Tx, y\rangle = \langle x, T^*y\rangle$$

for all $x, y \in H$ and $T \in B(H)$.

(iv) Let G be a unimodular locally compact group and $\mathcal{A}$ the convolution algebra $L^1(G)$ for each $f \in L^1(G)$, put $f^*(s) = f(s^{-1})$ where $s \in G$ with the map $f \to f^*$, then $\mathcal{A}$ is an involutive algebra and hence C*-algebra.

1.7.4: Positive linear functional and the GNS construction. See[20].

A self adjoint element a in a C*-algebra $\mathcal{A}$ is said to be positive if $\sigma(a) \subset [0, \infty)$ and we write $a \geq 0$. The set of positive elements of $\mathcal{A}$ is denoted by

$$\mathcal{A}^+ = \{a \in \mathcal{A} : a \geq 0\}$$

or equivalently,

$$\mathcal{A}^+ = \{a^*a : a \in \mathcal{A}\}.$$

Let $\mathcal{A}$ and $\mathcal{B}$ be C*-algebras. A linear map $f : \mathcal{A} \to \mathcal{B}$ is said to be positive if

$$f(\mathcal{A}^+) \subset \mathcal{B}^+.$$

A positive map is said to be faithful if $a \geq 0$ and $f(a) = 0$ implies that $a = 0$.
Any *- homomorphism is positive. Thus every *-homomorphism ϕ is faithful exactly when it is injective; $\phi(a) = 0$ if and only if $\phi(a^*a) = 0$.
Now for all $\mathcal{B} \subset \mathbb{C}$, the linear functional $f : \mathcal{A} \to \mathbb{C}$ is positive since $f(a^*a) \geq 0$ for all $a \in \mathcal{A}$ and is called **a positive linear functional**. For example, if $T \in B(H)$ and $x \in H$, then $f(T) = \langle Tx, x\rangle$ is a positive linear functional on $B(H)$ since

$$f(T^*T) = \langle T^*Tx, x\rangle = \langle Tx, Tx\rangle = \|Tx\|^2 \geq 0.$$

A positive linear functional f on $\mathcal{A}$ of norm one is called a **state** on $\mathcal{A}$. The set of all states of $\mathcal{A}$ denoted by $S(\mathcal{A})$ is called a **state space**. $S(\mathcal{A})$ is non-empty since by Hahnn Banach theorem, there exist $f \in \mathcal{A}^*$ such that $f(e) = 1 = \|f\|$.

The state space $S(\mathcal{A})$ forms a compact subset of $\mathcal{A}^*$ in weak*-topology; $S(\mathcal{A})$ is closed and convex and

$$S(\mathcal{A}) = \bigcap_{a\in\mathcal{A}} \{f \in \mathcal{A}^* : f(a) \in [0,\infty)\}.$$

Some of the examples of state include;

(a) Let $\mathcal{A}$ be a C*-algebra and $\phi : \mathcal{A} \to B(H)$ be a $*$-homomorphism. For all $x \in H$, we define a map $\psi_x : \mathcal{A} \to \mathbb{C}$ by

$$\psi_x(a) = \langle \phi(a)x, x\rangle$$

for all $a \in \mathcal{A}$.

Then ψ_x is a positive linear functional on $\mathcal{A}$ of norm 1 and hence a state.

(b) Let Ω be a compact Hausdorff space and μ the probability measure on Ω. We define $\psi : C(\Omega) \to \mathbb{C}$ by

$$\psi(f) = \int_\Omega f(x)d\mu(x)$$

for all $f \in C(\Omega)$, then ψ is a state.

(c) Let x be a vector in a Hilbert space H. Define $f : B(H) \to \mathbb{C}$ such that $T \to \langle Tx, x\rangle$ for $T \in B(H)$. Then f is a positive linear functional on $B(H)$. If x is a unit vector, then f is a state on $B(H)$.

A **representation** of a C*-algebra is the pair (π, H) where H is a Hilbert space and $\pi : \mathcal{A} \to B(H)$ is a *-homomorphism.
A representation is said to be **faithful** if π is injective.

Theorem 1.7.4.1:
If a is a normal element of a non-zero C-algebra $\mathcal{A}$, then there is a state f of $\mathcal{A}$ such that $\|a\| = |f(a)|$.*
See [20] for the proof.

Theorem 1.6.4.2:(Gelfand-Naimark Segal).

If $\mathcal{A}$ is a C*-algebra, then there exist a Hilbert space H and a universal faithful representation $\pi : \mathcal{A} \to B(H)$.

For the proof see [20].

1.7.5: Elementary operator.

There are various settings for the definition of the elementary operators. The elementary operator can be defined on separable infinite dimensional complex Hilbert space H, normed space, Banach space or on C^*- algebras.

Let H be a complex Hilbert space and $B(H)$ the algebra of all bounded linear operators on H. For a single operator $A \in B(H)$, we define two elementary operators; L_A and R_A on $B(H)$ called the left multiplication operator and right multiplication operator respectively given by:

$$L_A(X) = AX \ \ and \ \ R_A(X) = XA$$

for every $X \in B(H)$.

Let $A = (A_1, ..., A_k)$ and $B = (B_1, ..., B_k)$ be two fixed $k-$ tuples of elements of $B(H)$ with $A_i, B_i \in B(H)$ for $1 \leq i \leq k$. The map $R_{AB} : B(H) \to B(H)$ is the general elementary operator induced by A and B given by

$$R_{AB}(X) = A_1XB_1 + ... + A_nXB_n = \sum_{i=1}^{k} A_iXB_i$$

for all $X \in B(H)$.

Other particular elementary operators for all $X \in B(H)$ are;

(a) The generalized derivation corresponding to A and B is $\delta_{AB}(X) = (L_A - R_B)X = AX - XB$.

(b) The inner derivation induced by A is $\delta_{AA}(X) = (L_A - R_A)(X) = AX - XA$.

(c) The basic elementary operator $M_{AB}(X) = L_AR_B(X) = AXB$.

(d) The Operator $\Delta_A(X) = (L_A + R_A)(X) = AX + XA$.

(e) The Jordan elementary operator $U_{AB}(X) = (M_{AB} + M_{BA})(X) = AXB + BXA$.

(f) The operator $V_{AB}(X) = (M_{AB} - M_{BA})(X) = AXB - BXA$.

The elementary operators are linear and bounded.

1.7.6: Finite rank operator.

Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces. A linear map $T : \mathcal{X} \to \mathcal{Y}$ has a finite rank if its range is finite dimensional. A finite rank operator need not to be bounded. We denote the set of finite rank operators $T \in B(\mathcal{X}, \mathcal{Y})$ by $\mathcal{B}_{oo}(\mathcal{X}, \mathcal{Y})$ which is clearly a vector space.
If $f \in \mathcal{X}^*$ and $y \in \mathcal{Y}$, we define an operator $y \otimes f : \mathcal{X} \to \mathcal{Y}$ by

$$y \otimes f(x) = f(x)y.$$

$y \otimes f : \mathcal{X} \to \mathcal{Y}$ is linear and

$$\begin{aligned} \|y \otimes f\| &= \sup\{\|(y \otimes f)(x)\| : \|x\| \leq 1\} \\ &= \sup\{\|f(x)y\| : \|x\| \leq 1\} \\ &\leq \|y\| \sup\{|f(x)| : \|x\| \leq 1\} \\ &= \|f\|\|y\|. \end{aligned}$$

Thus $y \otimes f$ is bounded.
$(y \otimes f)(H) \subseteq span\{y\}$, so $y \otimes f$ has a finite rank therefore, $y \otimes f \in \mathcal{B}_{oo}(\mathcal{X}, \mathcal{Y})$.

The following theorems give representation for bounded finite rank operators. See [16]

Theorem 1.7.6.1:
If $T \in \mathcal{B}_{oo}(\mathcal{X}, \mathcal{Y})$ and $u_1, ..., u_n$ is a basis for $T(\mathcal{X})$, then there are unique $f_1, ..., f_n \in \mathcal{X}^*$ such that $T = \sum_{i=1}^{n} u_i \otimes f_i$.

Proof.

Let $u_1, ..., u_n$ be a basis for $T(\mathcal{X})$.

Now, $T(\mathcal{X}) \subseteq \mathcal{Y}$ since $T : \mathcal{X} \to \mathcal{Y}$. and so $u_1, ..., u_n \in \mathcal{Y}$.

Given unique $f_1, ..., f_n \in \mathcal{X}^*$, then for $T \in \mathcal{B}_{oo}(\mathcal{X}, \mathcal{Y})$ we have by the definition of finite rank operator that

$u_1 \otimes f_1 + u_2 \otimes f_2 + ... + u_n \otimes f_n : \mathcal{X} \to \mathcal{Y}$

This implies that $\sum_{i=1}^{n} u_i \otimes f_i : \mathcal{X} \to \mathcal{Y}$.

Thus $T = \sum_{i=1}^{n} u_i \otimes f_i$. $\square$

Theorem 1.7.6.2:

If there are $u_1, ..., u_n \in \mathcal{Y}$ and $f_1, ..., f_n \in \mathcal{X}^*$ such that $T = \sum_{i=1}^{n} u_i \otimes f_i$, then $T^* = \sum_{i=1}^{n} f_i \otimes u_i$.

Proof.

For $y \in \mathcal{Y}$, we define $F_y : \mathcal{Y}^* \to \mathbb{C}$ by $F_y(\lambda) = \lambda(y)$. So $F_y \in (\mathcal{Y}^*)^*$ and we write $y = T_y$.

Let $y \in \mathcal{Y}$ and $f \in \mathcal{X}^*$. If $x \in \mathcal{X}$ and $g \in \mathcal{Y}^*$ then

$\langle (y \otimes f)x, g \rangle = f(x)\langle y, g \rangle = f(x)g(y)$ where

$\langle ., . \rangle : \mathcal{Y} \times \mathcal{Y}^*$ is the dual pairing.

But,

$f(x)g(y) = \langle x, g(y)f \rangle = \langle x, F_y(g)f \rangle = \langle x, (F_y \otimes f)(g) \rangle$ where,

$\langle ., . \rangle : \mathcal{X} \times \mathcal{X}^* \to \mathbb{C}$ is the dual pairing.

We have $F_y \otimes f \in B(\mathcal{Y}^*, \mathcal{X}^*)$ and hence

$(y \otimes f)^* = F_y \otimes f = y \otimes f$.

This shows that the adjoint of each term $u_i \otimes f_i$ in T is $f_i \otimes u_i$ and the adjoint of the sum is the sum of adjoints of the terms. Thus, if $T \in \mathcal{B}_{oo}(\mathcal{X}, \mathcal{Y})$ then $T^* \in \mathcal{B}_{oo}(\mathcal{Y}^*, \mathcal{X}^*)$. $\square$

Theorem 1.7.6.3:

If $\mathcal{X}$ is a Banach space, then $\mathcal{B}_{oo}(\mathcal{X})$ is a two sided ideal in the Banach algebra $B(\mathcal{X})$.

Proof.

We recall that $\mathcal{B}_{oo}(\mathcal{X},\mathcal{Y})$ is a vector space. Since here, $\mathcal{Y}=\mathcal{X}$, then $\mathcal{B}_{oo}(\mathcal{X})$ is a vector space.

If $A\in\mathcal{B}_{oo}(\mathcal{X})$ and $T\in B(\mathcal{X})$, then $AT\in B(\mathcal{X})$ and $A(T(\mathcal{X}))\subseteq A\mathcal{X}$, which is finite dimensional. So $AT\in\mathcal{B}_{oo}(\mathcal{X})$.

$TA\in B(\mathcal{X})$ and $T(A(\mathcal{X}))$ is the image of a finite dimensional subspace under T, and so is itself finite dimensional. Hence $TA\in\mathcal{B}_{oo}(\mathcal{X})$. □

We now proceed to show that finite rank operators are compact.

If $T:\mathcal{X}\to\mathcal{Y}$ is a linear map and $\mathcal{U}$ the open unit ball in $\mathcal{X}$, then T is compact if $\overline{T(\mathcal{U})}$ in $\mathcal{Y}$ is compact. In other words, T is compact if $T(\mathcal{U})$ is totally bounded.

A convenient characterization of a compact operator is as follows;

A linear map $T:\mathcal{X}\to\mathcal{Y}$ is compact if and only if for every bounded sequence $x_n\in\mathcal{X}$ there is a subsequence x_{a_n} such that Tx_{a_n} converges in $\mathcal{Y}$.

We denote the set of compact operators by $\mathcal{B}_o(\mathcal{X},\mathcal{Y})$ which is a vector space.

For $T\in B(\mathcal{X},\mathcal{Y})$, it is clear that $T\in\mathcal{B}_o(\mathcal{X},\mathcal{Y})$ if and only if $T^*\in\mathcal{B}_o(\mathcal{Y}^*,\mathcal{X}^*)$. Also, $B(\mathcal{X},\mathcal{Y})$ being a Banach space implies that $\mathcal{B}_o(\mathcal{X},\mathcal{Y})$ is a Banach space as well with the operator norm.

The following theorem shows that a bounded finite rank operator is a compact operator. Since a limit of compact operators is a compact operator, then it follows from this that a limit of a bounded finite rank operator is a compact operator.

Theorem 1.7.6.4:

If $T\in\mathcal{B}_{oo}(\mathcal{X},\mathcal{Y})$ then $T\in\mathcal{B}_o(\mathcal{X},\mathcal{Y})$.

Proof.

Let $\mathcal{U}$ be the open unit ball in $\mathcal{X}$. Since T is bounded and $\mathcal{U}$ is a bounded set in $\mathcal{X}$, $T(\mathcal{U})$ is a bounded set in $\mathcal{Y}$. But $T(\mathcal{X})$ is a finite dimensional vector space and hence, the closure of $T(\mathcal{U})$ in $T(\mathcal{X})$ is a compact subset of $T(\mathcal{X})$. $T(\mathcal{X})$ is finite dimensional so it is a closed subset of $\mathcal{Y}$. Thus the closure of $T(\mathcal{U})$ in $\mathcal{Y}$ is a compact subset of $\mathcal{Y}$. □

If H is a Hilbert space then $B(H)$ is a C*-algebra (as the adjoint of $T \in B(H)$ is not just an element of $B(H^*)$ but can be identified with an element of $B(H)$) and therefore the above theorems implies $\mathcal{B}_{oo}(H)$ and $\mathcal{B}_o(H)$ are two sided *-ideals in the C*-algebra $B(H)$.

1.7.7: Trace class operators

Let $T \in B(H)$. We define $|T|$ to be the unique operator $S \in B(H)$ with $S \geq 0$ such that $S^2 = T^*T$.
If $\{e_i : i \in I\}$ is an orthonormal basis for H, we say that $T \in B(H)$ is trace class if $\sum_{i\in I}\langle |T|e_i, e_i\rangle < \infty$. We denote the set of trace class operators in $B(H)$ by $B_1(H)$. The trace class operator norm is defined by

$$\|T\|_1 = \sum_{i\in I}\langle |T|e_i, e_i\rangle.$$

If $T \in B_1(H)$ and ε is an orthonormal basis for H, we define the trace of T written as $tr\ T$, to be

$$tr\ T = \sum_{e\in\varepsilon}\langle Te, e\rangle.$$

Since $tr : B_1(H) \to \mathbb{C}$ is a positive linear functional that is, tr is a linear functional from $B_1(H)$ to $\mathbb{C}$. If $T \in B_1(H)$ is a positive operator, then trT is real and ≥ 0. If $T \in B_1(H)$ is a positive operator, then it is diagonalizable and since being a bounded trace class operator implies that it is compact, there is an orthonormal basis $\{e_i : i \in I\}$ for H such that $T = \sum_{i\in I}\langle Te_i, e_i\rangle e_i \otimes e_i$, where the series converges in the strong operator topology.
Since T is positive, $\langle Te_i, e_i\rangle$ is a real nonnegative number for each $i \in I$.
$trT = 0$ means that $\sum_{i\in I}\langle Te_i, e_i\rangle = 0$ and because this is a series of nonnegative terms, they must be all 0.
Substituting this into the expression of T gives $T = 0$, showing that $tr : B_1(H) \to \mathbb{C}$ is a non-negative definite linear functional.

1.7.8: Convex set

A set P is said to be convex if the line segment between any two points in P lies in P, that is, if $x, y \in P$ then $z = tx + (1-t)y \in P$ for all $t \in [0, 1]$.

Given any non empty set P, there is a smallest convex set containing P denoted by $con(P)$ and is referred to as the convex hull of P. Equivalently, it is the intersection of all convex sets containing P.

1.7.9: Generalizations of the Numerical Range.

For operators on Hilbert space H, the notion of numerical range (or field of values) is important in various applications in the study of operators. Here we introduce the numerical range on Hilbert space.

Let T be an operator in $B(H)$, the algebra of all bounded operators on H. The numerical range $W(T)$ of an operator T is a subset of the complex plane $\mathbb{C}$ defined by

$$W(T) = \{\langle Tx, x\rangle : \ x \in H, \ \ \|x\| = 1\}.$$

It is known that the numerical range is convex, that is, if $\lambda_1, \lambda_2 \in W(T)$ then $\lambda = t\lambda_1 + (1-t)\lambda_2 \in W(T)$ for every real number $0 \leq t \leq 1$.

Some of the basic properties of numerical range are;

(i) $W(T^*) = \overline{W(T)}$,

(ii) $\overline{W(T)}$ contains the spectrum of T,

(ii) If $\mu, \lambda \in \mathbb{C}$ then $W(\lambda T + \mu I_H) = \lambda W(T) + \mu$,

(iv) $W(U^*TU) = W(T)$ for all unitary operators U and

(v) $W(T+S) \subseteq W(T) + W(S)$ for all $T, S \in B(H)$.

For the proofs of these properties, see [6], [11] and [15].

Other numerical ranges include:

(a) **Maximal numerical range** of T defined by the set

$$W_o(T) = \{\lambda : \ \langle Tx_n, x_n\rangle \to \lambda \, where \, \|x_n\| = 1 \, and \, \|Tx_n\| \to \|T\|\}$$

where x_n is a sequence in H and $\lambda \in \mathbb{C}$.

Stampfli [28] introduced this numerical range and showed that it is nonempty, closed, convex and is contained in the closure of the classical numerical range.

Lemma 1.7.9.1:

Let H be a complex Hilbert space and $B(H)$ the algebra of all bounded operators on H. If $\|T\| = \|x\| = 1$ and $\|Tx\|^2 \geq (1-\varepsilon)$, then
$\|(T^*T - I)x\|^2 \leq 2\varepsilon$.

See Stampfli [28] for the proof.

Lemma 1.7.9.2: (Stampfli [28])

Let H be a complex Hilbert space and $B(H)$ the algebra of all bounded operators on H. Then for all $T \in B(H)$, the set $W_o(T)$ is nonempty,closed, convex and contained in the closure of the numerical range.

Proof.

We only show the convexity since it is clear that the maximal numerical range $W_o(T)$ is nonempty and closed.

Let λ, $\mu \in W_o(T)$ and $x_n, y_n \in H$. Assume without loss of generality that $\|T\| = 1$. Assume also that
$\|x_n\| = \|y_n\| = 1$, $\langle Tx_n, x_n\rangle \to \lambda$ and $\langle Ty_n, y_n\rangle \to \mu$.

Consider $T_n = P_nTP_n$, where P_n is the projection on H of $\{x_n, y_n\}$.

Let η be a point on the line segment joining λ and μ. Then for each n it is possible, by Toeplitz-Hausdorff Theorem, to choose α_n, β_n such that $\langle Tu_n, u_n\rangle = \langle T_nu_n, u_n\rangle \to \eta$ and $\|u_n\| = 1$, where $u_n = \alpha_nx_n + \beta_ny_n$.

Note that $|\langle x_n, y_n\rangle| \leq 0 < 1$ for n sufficiently large; that is, the angle between x_n and y_n is bounded away from 0. (It is difficult to compute an explicit upper bound for $lim\ sup|\langle x_n, y_n\rangle|$ in terms of λ and μ). Thus, there exist a constant M such that $|\alpha_n| \leq M$ and $\|\beta_n| \leq M$ for large n, where $\|\alpha_nx_n + \beta_ny_n\| = 1$. Since by lemma 1.7.8.1 above, $\|Tu_n\| = \langle T^*Tu_n, u_n\rangle = \|u_n\|^2 - 2M\varepsilon_n$ where $\varepsilon_n \to 0$, and thus it follows that $\|Tu_n\| \to 1$. Since $\langle Tu_n, u_n\rangle \to \eta$ this completes the proof. $\square$

Stampfli [28] used the maximal numerical range to determine the norm of inner derivation acting on the Banach algebra $B(H)$ as

$$\|\delta_T\| = \inf\{2\|T - \lambda I\| : \lambda \in \mathbb{C}\}.$$

Theorem 1.7.9.3:(Stampfli [28])

Let δ_T be a derivation on $B(H)$. Then, $\|\delta_T\| = \inf\{2\|T - \lambda\| : \lambda \in \mathbb{C}.\}$

Proof.

Since

$$\begin{aligned}\|TX - XT\| &= \|(T - \lambda)X - X(T - \lambda)\| \\ &\leq 2\|T - \lambda\|\|X\|.\end{aligned}$$

It follows that $\|\delta_T\| \leq \inf\{2\|T - \lambda\| : \lambda \in \mathbb{C}\}$.

On the other hand,

$\|T-\lambda\|$ is large for λ large, so $\inf \|T-\lambda\|$ must be taken on at some point, say z_o. But $\|T - z_o\| \leq \|(T - z_o) + \lambda\|$ for all $\lambda \in \mathbb{C}$ implies that $0 \in W_o(T - z_o)$. Hence $\|\delta_T\| = \|\delta_{(T-z_o}\| = 2\|T - z_o\|$ which completes the proof. □

(b) If $\mathcal{A}$ is a C*-Banach algebra with identity e, $a \in \mathcal{A}$ and $S(\mathcal{A}) = \{f \in \mathcal{A}^* : \ f(e) = 1 = \|f\|\}$, the set of states on $\mathcal{A}$ then the **algebraic numerical range** of $a \in \mathcal{A}$ is the set

$$V(a/\mathcal{A}) = \{f(a) : \ f \in S(\mathcal{A})\}$$

The set $V(a)$ is known to be non empty, convex and compact. This follows immediately from the corresponding properties of the set of states being convex and weak* compact in $\mathcal{A}^*$. Since the map $f \to f(x)$ is weak* continuous on $\mathcal{A}$ then the range is compact and convex. See [27] for details.

It has also been shown that for $\mathcal{A} = B(H)$ the algebraic numerical range is equal to the closure of the classical numerical range that is, $V(T) = \overline{W(T)}$ for $T \in B(H)$. See[1].

The algebraic numerical range of the basic elementary operator M_{AB} is defined by

$$V(M_{AB/B(H)}) = \{f(M_{AB}) : \ f \in B(B(H))^*, \ \|f\| = 1 = f(I)\}$$

(c) Let $\mathcal{X}$ be a complex Banach space with dual space $\mathcal{X}^*$ and $B(\mathcal{X})$ the complex Banach algebra of all bounded linear operators on $\mathcal{X}$. For the operator $T \in B(\mathcal{X})$, the **spatial numerical range** $V_o(T)$ of T is defined by

$$V_o(T) = \{f(Tx) : (x, f) \in \prod\}$$

where $\prod = \{(x, f) \in \mathcal{X} \times \mathcal{X}^* : \|x\| = 1 = \|f\| = f(x)\}$.
It has been shown that the closure of the convex hull of the spatial numerical range is equal to the usual classical numerical range.
When $\mathcal{X}$ is a Hilbert space, $\|x\| = \|x^*\| = \langle x, x^* \rangle$ if and only if x^* is a function given by $x^*y = \langle y, x \rangle, y \in \mathcal{X}$ thus $V(T)$ in this case coincides with $W(T)$. See[19] and [27].

(d) Let H be a Hilbert space and $B(H)$ the algebra of all bounded linear operators on H. Then for n-tuple $T = (T_1, T_2, ..., T_n)$ of self adjoint operators on H, the **joint numerical range** of $T \in B(H)$ is defined as

$$W_J(T) = \{(\langle T_1x, x\rangle, \langle T_2x, x\rangle, ..., \langle T_nx, x\rangle) : x \in H, \|x\| = 1\}$$

(e) We denote $K(\mathcal{X})$ the ideal of all compact operators acting on a complex Banach space $\mathcal{X}$ and let π be the canonical projection from $B(\mathcal{X})$ onto Calkin algebra $B(\mathcal{X})/K(\mathcal{X})$. Denote further by $\|.\|_e$ the essential norm $\|T\|_e = \inf\{\|T + K\| : K \in K(\mathcal{X})\}$. Let $\mathcal{X}$ be an infinite-dimensional Banach space and $T \in B(\mathcal{X})$. The Essential numerical range $V_e(T)$ of T is defined by

$$V_e(T) = V(\pi(T), B(\mathcal{X})/K(\mathcal{X}), \|.\|_e)$$

1.7.10: Spectrum, Spectral radius and numerical radius

Let $B(\mathcal{X})$ be a complex unital algebra with identity I and let $A \in B(\mathcal{X})$ where $\mathcal{X}$ is a complex normed space. The spectrum of A denoted by $\sigma(A)$ is the set of all $\lambda \in \mathbb{C}$ such that $(A - \lambda I)$ has no inverse, that is,

$$\sigma(A) = \{\lambda \in \mathbb{C} : A - \lambda I \ \ is \ \ not \ \ invertible\}.$$

It is known that the closure $\overline{W(A)}$ of $W(A)$ contains the spectrum of A. The spectral radius of A is defined to be the number

$$r(A) = \sup\{|\lambda| : \lambda \in \sigma(A)\}.$$

Since the spectrum of any $A \in B(\mathcal{X})$ is non-empty and compact we have that the spectral radius $r(A)$ is the smallest number r such that the disk $\{\lambda : |\lambda| \leq r\}$ contains the spectrum of A that is, $r(A) = \sup\{|\lambda| : \lambda \in \sigma(A)\} = \lim_{k\to\infty} \|A^k\|^{\frac{1}{k}}$. The numerical radius $w(A)$ of A is given by

$$w(A) = \sup\{|\lambda| : \lambda \in W(A)\}.$$

The numerical radius $w(A)$ is a norm equivalent to the operator norm $\|A\|$ which satisfies $\frac{1}{2}\|A\| \leq w(A) \leq \|A\|$.

CHAPTER TWO

LITERATURE REVIEW

The literature related to elementary operators is by now very large, and there are excellent surveys on expositions of certain aspects. Elementary operators first appeared in a series of notes by Sylvester in 1880's in which he computed the eigenvalues of the matrix operators corresponding to the elementary operator $R_{A,B}$ on the set of square matrices. The term elementary operator was coined by Lumer and Rosenblum [18] (in a more general Banach algebra context) where they computed the spectra of such operators and gave their applications to systems of operator equations in 1959.

The structural theory of elementary operators has been an interesting area of research mainly on the norm and numerical range of these operators. Over time, the relationship between spatial numerical ranges, numerical ranges and their spectra has been investigated. Most importantly, it has been shown that the numerical range is convex.

The concept of numerical range of operators was introduced by Toeplitz in 1918 [31] for matrices, a concept easily extensible to bounded linear operators on a Hilbert space. He proved that the numerical range is a convex set by the classical Toeplitz theorem which is an important property of numerical range.

In 1961, this concept was independently extended by Lumer [19] to bounded linear operators acting on arbitrary Banach space by introducing the spatial numerical range of an operator on a Banach space.

In 1968, Stampfli and Williams [27] showed that the closure of the numerical range is equal to the algebraic numerical range, that is, $V(A/B(H)) = \overline{W(A)}$ for an operator $A \in B(H)$. Later in 1970, Stampfli [28] introduced the concept of maximal numerical range of bounded linear operator. He established that the maximal numerical range as a set is non empty, closed and convex and used it to derive the norm of the inner and generalized derivation. He showed that $\|\delta_A\| = 2 \inf\{\|A - \lambda\| : \lambda \in \mathbb{C}\}$ and $\|\delta_A\| = 2\|A\|$ if and only if $0 \in W_o(A)$.

Bonsal and Duncan [7] in 1973 on their research give detailed generalization of the numerical range for Banach space and Banach algebra setting. They defined the algebraic numerical range on a complex unital Banach algebra $\mathcal{A}$ as
$V(a/\mathcal{A}) = \{f(a) : \; f \in S(\mathcal{A})\}$, where $S(\mathcal{A})$ is the set of states on $\mathcal{A}$.

Kyle [17] in 1978 using the known results on spectra of inner derivation examined the relationship between the numerical range of a derivation and that of its implementing operator on a complex unital Banach algebra. He established that the algebraic numerical range of the a derivation is equal to the sum of the algebraic numerical ranges of the implementing operators, that is,
$V(\mathfrak{T}_{AB/B(B(H))}) = V(A/B(H)) + V(B/B(H))$.

On his part, Shaw [26] in 1984 working on normed linear spaces $\mathcal{Y}$ and $\mathcal{X}$, proved that the algebraic numerical range of a generalized derivation restricted to δ the subspace of $B(\mathcal{Y}, \mathcal{X})$ is equal to the difference of the algebraic numerical range of the implementing operators A and B, that is, $V(\delta_{AB/B(\delta)}) = V(A/B(\mathcal{Y})) - V(B/B(\mathcal{X}))$. Here, $B(\mathcal{Y}, \mathcal{X})$ is the space of all operators from $\mathcal{Y}$ to $\mathcal{X}$, $B(\mathcal{Y})$ the algebra of all bounded operators on $\mathcal{Y}$ and $B(\mathcal{X})$ the algebra of all bounded linear operators on $\mathcal{X}$.

Seddik [23] in 2001 established that the algebraic numerical range of a generalized derivation equals to that of the same derivation when restricted to the Banach space of p-Schatten class of operators on H. Furthermore, he showed that these numerical ranges are equal to the difference in the algebraic numerical ranges of the implementing operators, that is, $V(\delta_{AB}) = V(\delta_{AB/\ell_p}) = V(A) - V(B)$. Using these results and working on complex unital Banach algebra, in 2002 Seddik [24] showed that the convex hull of the joint numerical range of elements implementing the general elementary operator is contained in the algebraic numerical range of the elementary operator, that is,

$$co\{\sum_{i=1}^{k}\alpha_i\beta_i : (\alpha_1,\dots,\alpha_2) \in W_J(A),\ (\beta_1,\dots,\beta_k) \in W_J(B)\}^- \subset V(R_{A,B}).$$

He further established that the inclusion is strict if R_{AB} is a multiplication operator M_{AB} induced by non scalar self adjoint operators and an equality if R_{AB} is taken to be a generalized derivation. Again in 2004, Seddik [25] showed that the result is the same when the elementary operator R_{AB} is restricted a norm ideal J of $B(\mathcal{X})$ the complex Banach algebra of all bounded linear operators on a complex Banach space $\mathcal{X}$.

In 2014, Barraa [4] expressed the algebraic numerical range of the general elementary operator R_{AB} in terms of the classical numerical range of the operators that implement it in operator algebra $B(H)$. He established that
$V(R_{AB/B(B(H))}) = [\bigcup_{U\in U(H)} W(\sum_{i=1}^{k} UA_iU^*B_i)]^-$ where $A = (A_1,...,A_k)$, $B = (B_1,...,B_k)$ are k-tuples of elements of $B(H)$ and $U(H)$ is the set of unitary operators. He extended this expression to the context of C*-algebra in 2015 [5] to give
$V(R_{ab/B(\mathcal{A})}) = [\bigcup\{V(\sum_{i=1}^{k} u^*a_iub_i,\ /\mathcal{A}) : u \in U\}]^-$.

In our study we determine the formula for numerical range of multiplication operator in which Barraa's work [4], [5] forms the basis of our research.

On the other hand, the norm of a generalized derivation can be traced back to Stampfli's work in 1970 [28]. In his work, he gave an elegant formula for the norm of a generalized derivation as $\|\delta_{AB}\| = \inf\{\|A-\lambda\| + \|B-\lambda\| : \lambda \in \mathbb{C}\}$ using maximal numerical range.

Fialkow [9] in 1979 and 1992 [10] estimated the norm of the generalized derivation restricted on the norm ideal J in $B(H)$ in the opposite direction.

In 2001, Barraa and Boumazgour [3] established that the norm of a generalized derivation restricted on the norm ideal is less or equal to the norm of the generalized derivation, i.e. $\|\delta_{JAB}\| \leq \|\delta_{AB}\|$. They further characterized the class of operators for which the equality holds. Boumazgour [8] in 2006 further established that for every pair (A, B) of operators on H, there exists a positive number α_i satisfying $1 \leq \alpha_i \leq 2$ such that $\|\delta_{AB}\| \leq \alpha_i \|\delta_{AB}\|$. He extended this work in 2006 where he compared the norm of a generalized derivation on a Hilbert space H with the norm of its restriction to Schatten norm ideals.

In our study we establish that the norm of the generalized inner derivation is equal to the summation of the norms of the fixed operators that implement it.

CHAPTER THREE

RESULTS AND DISCUSSION

In this chapter we give results on some properties of the numerical range of the basic elementary operator. In particular, we determine the relationship between the algebraic numerical range of the basic elementary operator and the classical numerical range of the implementing operators. We also determine the norm of a generalized derivation.

3.1 Algebraic numerical range of the basic elementary operator

In this section, we show that some properties of the numerical range of an operator in a Hilbert space holds for the algebraic numerical range of the basic elementary operator. In particular, we prove that the algebraic numerical range of the basic elementary operator is equal to the classical numerical range of the implementing operators in the operator algebra $B(H)$ and extend the relation to when $B(H)$ is a C*-algebra.

Let $\mathcal{A}$ be a unital Banach algebra. The basic elementary operator
$M_{a,b} : \mathcal{A} \to \mathcal{A}$ is defined by

$$M_{ab}(x) = axb \tag{3.1.1}$$

where $x \in \mathcal{A}$ and $a, b \in \mathcal{A}$ are fixed.

For $\mathcal{A} = B(H)$ a C*-algebra the basic elementary operator
$M_{AB} : B(H) \to B(H)$ is defined by;

$$M_{AB}(X) = AXB \tag{3.1.2}$$

for all $X \in B(H)$.

We recall that the algebraic numerical range of the basic elementary operator M_{AB} is given by

$$V(M_{AB/B(H)}) = \{f(M_{AB}) : \ f \in B(B(H))^*, \ \|f\| = 1 = f(I)\}.$$

Proposition 3.1.1

Let the map $M_{AB} : B(H) \to B(H)$ be the basic elementary operator. Then we have that

(i) $V(\alpha M_{AB} + \beta I) = \alpha V(M_{AB}) + \beta$,

(ii) $V(U^* M_{AB} U) = V(M_{AB})$,

(iii) $V(M_{A_1B_1} + M_{A_2B_2}) \subseteq V(M_{A_1B_1}) + V(M_{A_2B_2})$ and

(iv) $\sigma(M_{AB}) \subseteq V(M_{AB})$ where $A, B \in B(H)$, $\alpha, \beta \in \mathbb{C}$, a unitary $U \in B(H)$ and the identity operator $I \in B(H)$.

Proof.

(i) By definition,

$$\begin{aligned} V(\alpha M_{AB} + \beta I) &= \{f(\alpha M_{AB} + \beta I) : f \in B(B(H))^*, \ \|f\| = 1 = f(I)\} \\ &= \{f(\alpha M_{AB}) + f(\beta I) : f \in B(B(H))^*, \ \|f\| = 1 = f(I)\} \\ &= \{\alpha f(M_{AB}) + \beta f(I) : f \in B(B(H))^*, \ \|f\| = 1 = f(I)\} \\ &= \alpha V(M_{AB}) + \beta. \end{aligned}$$

(ii) For a unitary $U \in B(H)$,

$$\begin{aligned} V(U^* M_{AB} U) &= \{f(U^* M_{AB} U) : f \in B(B(H))^*, \ \|f\| = 1 = f(I)\} \\ &= \{f(M_{AB} U^* U) : f \in B(B(H))^*, \ \|f\| = 1 = f(I)\} \\ &= \{f(M_{AB}) f(U^* U) : f \in B(B(H))^*, \ \|f\| = 1 = f(I)\} \\ &= \{f(M_{AB}) : f \in B(B(H))^*, \ \|f\| = 1 = f(I)\} \\ &= V(M_{AB}). \end{aligned}$$

(iii) For $M_{A_1B_1}, M_{A_2B_2} \in B(B(H))$, we have
$V(M_{A_1B_1} + M_{A_2B_2}) = \{f(M_{A_1B_1} + M_{A_2B_2}) : f \in B(B(H))^*,\ \|f\| = 1 = f(I)\}$
$\subseteq \{f(M_{A_1B_1}) : f \in B(B(H))^*,\ \|f\| = 1 = f(I)\} + \{f(M_{A_2B_2}) : f \in B(B(H))^*,\ \|f\| = 1 = f(I)\}$
$= V(M_{A_1B_1}) + V(M_{A_2B_2})$

(iv) Since both the spectrum and the numerical range transform properly under affine mappings of operators, it is enough to prove that if $0 \in \sigma(M_{AB})$ then $0 \in \overline{W(M_{AB)}}$.
Let $0 \in \sigma(M_{AB})$ that is, M_{AB} is not invertible. Then there are two possibilities, either M_{AB} is not bounded below or M_{AB} is bounded below but is not onto.
For the first possibility, if M_{AB} is not bounded below, then there exists unit vectors $x_n \in B(H)$ such that $\langle M_{AB}x_n, x_n\rangle \to 0$.
Thus $\lim_{n\to\infty}\langle M_{AB}x_n, x_n\rangle = 0$.
Therefore $0 \in \overline{W(M_{AB)}}$ and since the algebraic numerical of an operator is equal to the closure of the classical numerical range of the operator then we have that $0 \in V(M_{AB})$.
For the second possibility, if M_{AB} is bounded below but not onto, then
$0 \neq (ranM_{AB})^{\perp} = Ker(M_{AB})^*$, hence $0 \in W(M_{AB})^*$ and therefore $0 \in W(M_{AB})$. □

In the following Theorem 3.1.2., we show that the algebraic numerical range of the basic elementary operator is a convex set.

Theorem 3.1.2

Let H be a Hilbert space and $B(H)$ the algebra of all bounded linear operators on H. Then the algebraic numerical range $V(M_{AB})$ is a convex set.

Proof.

We need to show that if $\alpha_1, \alpha_2 \in V(M_{AB})$ and $t \in (0, 1)$ then
$\alpha = t\alpha_1 + (1-t)\alpha_2 \in V(M_{AB})$ where $\alpha \in \mathbb{C}$.
Let $\alpha_1, \alpha_2 \in V(M_{AB})$ then there exist support functionals f_1 and$f_2 \in B(B(H))^*$ such that
$\alpha_1 = f_1(M_{AB}(X))$ and $\alpha_2 = f_2(M_{AB}(X))$ where $M_{AB} \in B(B(H))$,

$f_1(AB) = 1 = \|f_1\|$ and $f_2(AB) = 1 = \|f_2\|$.

We define f on $B(B(H))$ by

$$f(M_{AB}(X)) = tf_1(M_{AB}(X)) + (1-t)f_2(M_{AB}(X)).$$

We show that f is a state.

We first show that f is linear.

Let $\mu_1, \mu_2 \in \mathbb{C}$ and $M_{AB} \in B(B(H))$ then for any $X \in B(H)$ we have that,

$$\begin{aligned} f(\mu_1(M_{AB}(X)) + \mu_2(M_{AB}(X))) &= tf_1(\mu_1(M_{AB}(X)) + \mu_2(M_{AB}(X))) + (1-t) \\ &\quad f_2(\mu_1(M_{A,B}(X)) + \mu_2(M_{AB}(X))) \\ &= [tf_1(\mu_1(M_{AB}(X))) + (1-t)f_2(\mu_1(M_{AB}(X)))] \\ &\quad + [tf_1(\mu_2(M_{AB}(X))) + (1-t)f_2(\mu_2(M_{A,B}(X)))] \\ &= [\mu_1(tf_1(M_{AB}(X))) + \mu_1((1-t)f_2(M_{AB}(X)))] \\ &\quad + [\mu_2(tf_1(M_{AB}(X))) + \mu_2((1-t)f_2(M_{AB}(X)))] \\ &= \mu_1[tf_1(M_{AB}) + (1-t)f_2(M_{AB}(X))] \\ &\quad + \mu_2[tf_1(M_{AB}(X)) + (1-t)f_2(M_{AB})] \\ &= \mu_1(f(M_{AB}(X))) + \mu_2(f(M_{AB}(X))). \end{aligned}$$

Hence f is linear.

Next we prove that f is positive

$f((AXB)^*AXB) = tf_1((AXB)^*AXB) + (1-t)f_2((AXB)^*AXB) \geq 0$ since $f_1((AXB)^*AXB) \geq 0$ and $f_2((AXB)^*AXB) \geq 0$.

Lastly, we prove that $\|f\| = 1$.

Since

$f(AB) = tf_1(AB)+(1-t)f_2(AB)$ with $f_1(AB) = 1 = \|f_1\|$ and $f_2(AB) = 1 = \|f_2\|$, then

$$\begin{aligned} |f(AB)| &= |tf_1(AB) + (1-t)f_2(AB)| \\ &\leq |tf_1(AB)| + |(1-t)f_2(AB)| \\ &\leq t\|f_1\|\|AB\| + (1-t)\|f_2\|\|AB\| \\ &= \|AB\| \ \Rightarrow \|f\| \leq 1. \end{aligned}$$

Also,

$f(AB) = tf_1(AB) + (1-t)f_2(AB) = 1$ and

$$\begin{aligned} f(I) &= tf_1(I) + (1-t)f_2(I) \\ &= t + (1-t) \\ &= 1. \end{aligned}$$

But,

$1 = |f(I)| \leq \|f\|\|I\| = \|f\|$ which implies that $\|f\| \geq 1$.

Thus f is a state on $B(B(H))$ and therefore, $f(M_{AB}) \in V(M_{AB})$ so $V(M_{AB})$ is convex. $\square$

If $\mathcal{A} = B(\mathcal{X})$ where $B(\mathcal{X})$ is the algebra of bounded linear operators on a normed space $\mathcal{X}$ and $T \in B(\mathcal{X})$, then we have the spatial numerical range of T defined by;

$$V_o(T) = \{f(Tx) : x \in \mathcal{X}, f \in \mathcal{X}^* \ \ with \ \ \|f\| = \|x\| = 1\} \tag{3.1.3}$$

If $\mathcal{X} = H$ then we have the classical numerical range $W(T)$ for any $T \in B(H)$ defined by;

$$W(T) = \{\langle Tx, x\rangle : x \in H, \|x\| = 1\} \tag{3.1.4}$$

which is convex but not closed and in general $V(T) = \overline{W(T)}$.

So we have the classical numerical range of the basic elementary operator to be

$$W(AXB) = \{\langle AXBx, x\rangle : x \in H, \ \|x\| = 1\}. \tag{3.1.5}$$

Theorem 3.1.3

$\overline{W(M_{AB})} \subseteq V(M_{AB})$

Proof.

Let $\alpha \in \overline{W(AXB)}$ then there exists a convergent sequence $\{x_n\}_{n\geq 1}$ of unit vectors in H such that $\lim_{n\to\infty}\langle AXBx_n, x_n\rangle = \alpha$. We define a functional f on $B(B(H))$ by

$$f(AXB) = \lim_{n\to\infty}\langle AXBx_n, x_n\rangle = \alpha.$$

We then show that f is a state.

First, f is linear since if $A_1X_1B_1,\ A_2X_2B_2 \in B(B(H))$ and $\lambda, \mu \in \mathbb{C}$ then,

$$\begin{aligned} f(\lambda(A_1X_1B_1) + \mu(A_2X_2B_2)) &= \lim_{n\to\infty} \langle(\lambda(A_1X_1B_1) + \mu(A_2X_2B_2))x_n, x_n\rangle \\ &= \lim_{n\to\infty} \langle\lambda(A_1X_1B_1)x_n, x_n\rangle + \lim_{n\to\infty} \langle\mu(A_2X_2B_2)x_n, x_n\rangle \\ &= \lambda \lim_{n\to\infty} \langle(A_1X_1B_1)x_n, x_n\rangle + \mu \lim_{n\to\infty} \langle(A_2X_2B_2)x_n, x_n\rangle \\ &= \lambda f(A_1X_1B_1) + \mu f(A_2X_2B_2). \end{aligned}$$

f is positive since

$$\begin{aligned} f((AXB)^*(AXB)) &= \lim_{n\to\infty} \langle((AXB)^*AXB)x_n, x_n\rangle \\ &= \lim_{n\to\infty} \langle AXBx_n, AXBx_n\rangle \\ &= \{\lim_{n\to\infty} \|AXBx_n\|\}^2 = \|AXB\|^2 \geq 0. \end{aligned}$$

Finally, we show that $\|f\| = 1$.

For $I \in B(B(H))$ we have that,

$f(I) = \lim_{n\to\infty}\langle Ix_n, x_n\rangle = \lim_{n\to\infty}\langle x_n, x_n\rangle = \{\lim_{n\to\infty} \|x_n\|\}^2 = 1.$

$$\begin{aligned} f(AXB)| &= |\lim_{n\to\infty} \langle AXBx_n, x_n\rangle| \\ &\leq \lim_{n\to\infty} \|AXBx_n\| \lim_{n\to\infty} \|x_n\| \\ &= \|AXB\|. \end{aligned}$$

so that $\|f\| \leq 1$.

Since $f(I) = 1$, then $\|f\| = f(I) = 1$ and $1 = \|f(I)\| \leq \|f\|\|I\| = \|f\|$ so that $\|f\| \geq 1$.

Therefore, $\alpha = f(AXB) \in V(M_{AB})$ and hence $\overline{W(AXB)} \subseteq V(M_{AB})$. $\square$

The following preliminary results will be used to prove our first result on the algebraic numerical range of the basic elementary operator.

Lemma 3.1.4

Let A and B be elements in $B(H)$. Then, $W(AB) \subset V(M_{AB/B(B(H))})$ where $W(AB) = \{\langle ABx, x\rangle :\ x \in H,\ \|x\| = 1\}$.

Proof.

Let $\alpha \in W(AB)$ then by definition of the classical numerical range, there exist $x \in H$ with $\|x\| = 1$ such that;

$\alpha = \langle ABx, x\rangle = tr(AB(x \otimes x))$ where tr(.) is a linear functional trace.

We now define a linear functional $\Psi_{x\otimes x}$ by

$$\Psi_{x\otimes x}(X) = tr(X(x \otimes x)) = \langle Xx, x\rangle$$

on $B(H)$.

The linear functional is bounded and its norm is equal to one that is;

$\|\Psi_{x\otimes x}\| = \|x \otimes x\| = 1.$

The functional $\Psi_{x\otimes x}$ is also a state since

$\Psi_{x\otimes x}(I) = tr(x \otimes x) = \langle x, x\rangle = \|x\|^2 = 1$ and

$\Psi_{x\otimes x}(X^*X) = tr(X^*X(x \otimes x)) = \langle X^*Xx, x\rangle = \langle Xx, Xx\rangle = \|Xx\|^2 \geq 0.$

So $\Psi_{x\otimes x}(M_{AB}(I_H)) \in V(M_{AB/B(B(H))})$ and we have that

$\Psi_{x\otimes x}(M_{AB}(I_H)) = \Psi_{x\otimes x}(AB) = tr(AB(x \otimes x)) = \langle ABx, x\rangle = \alpha.$

Thus $W(AB) \subset W(M_{AB}) \subset V(M_{AB/B(B(H))})$. $\square$

We recall that the algebraic numerical range of an operator $a \in \mathcal{A}$ is given by

$$V(a/\mathcal{A}) = \{f(a) : \ f \in \mathcal{A}^* \ \|f\| = 1 = f(e)\}.$$

However, the following Theorem 3.1.5 also gives another expression for algebraic numerical range.

Theorem 3.1.5

Let $\mathcal{A}$ be a Banach algebra, then for any $a \in \mathcal{A}$;

$$V(a/\mathcal{A}) = \bigcap_{z\in\mathbb{C}} \{\lambda : |\lambda - z| \leq \|a - z\|\}.$$

See [27] for the proof.

The algebraic numerical range of the basic elementary operator can also be expressed in a similar manner as shown by the following results.

In the following Lemma 3.1.6, the distance between a point λ and a set S in the complex plane is denoted as $d(\lambda, S)$.

Lemma 3.1.6

If λ is not in $V(M_{AB})$, then $\|(M_{AB}-\lambda)^{-1}\| \leq d(\lambda, V(M_{AB}))^{-1}$, where $d(\lambda, V(M_{A,B}))$ is the distance from the point λ to the set $V(M_{AB})$ and $d(\lambda, V(M_{AB}))^{-1}$ its inverse.

Proof.

If λ is not in $V(M_{AB})$, then $(M_{AB} - \lambda)^{-1}$ exits since $V(M_{AB})$ is known to be a closed convex set that contains the spectrum $\sigma(M_{AB})$. Therefore we only need to show that $d[\lambda, V(M_{AB})]\|y\| \leq \|(M_{AB} - \lambda)y\|$ for all $y \in H$.

Now, for any $y \in H$ with $\|y\| = 1$, we choose a functional $g \in B(H)^*$ such that $\|g\| = 1 = g(y)$. Let $f(x) = g(xy)$ for $x \in H$. Then f is a state and

$$\begin{aligned} d[\lambda, V(M_{AB})]\|y\| &\leq |\lambda - f(M_{AB})| \\ &= |f(\lambda - M_{AB})| \\ &= |g(\lambda - M_{AB})y| \\ &\leq \|g\|\|(\lambda - M_{AB})y\| \\ &= \|(\lambda - M_{A,B})y\|. \square \end{aligned}$$

Theorem 3.1.7

If L is a closed convex subset of the plane, then $V(M_{AB}) \subset L$ if and only if $\|(M_{AB} - \lambda)^{-1}\| \leq d[\lambda, L]^{-1}$ for λ not in L.

Proof.

Assume that $V(M_{AB}) \subset L$. By lemma 3.1.6 we have that,
$\|(M_{AB} - \lambda)^{-1}\| \leq d(\lambda, V(M_{AB}))^{-1} \leq d(\lambda, L)^{-1}$ for λ which is not in L.
Conversely,
let $\|(M_{AB} - \lambda)^{-1}\| \leq d(\lambda, L)^{-1}$ for λ not in L. Then to show that $V(M_{AB}) \subset L$, we only need to show that every half plane M which contains L also contains $V(M_{AB})$.
Let M be the right-half plane, then $Rez \geq 0$ since $M \subset L$ then
$\|(1 + \varepsilon M_{AB})^{-1}\| \leq 1$ for all $\varepsilon > 0$.
If f is a state then we have that

$Ref((1+\varepsilon M_{AB})^{-1} \leq \|f\| \|(1+\varepsilon M_{AB})^{-1}\| \leq f(1)$. So,

$0 \leq f(1) - Ref(1+\varepsilon M_{AB})^{-1} = Ref(1-(1+\varepsilon M_{AB})^{-1})$.

Therefore $0 \leq Ref(\varepsilon M_{AB}(1+\varepsilon M_{AB})^{-1})$.

Dividing by ε and letting $\varepsilon \to 0$ we obtain that

$0 \leq Ref(M_{AB})$. Since f was arbitrary, then $V(M_{AB}) \subset L$. $\square$

Corollary 3.1.8

$\|(M_{AB}-\lambda)^{-1}\| \leq d(\lambda, L)^{-1} + o(1)$ *as* $\lambda \to \infty$.

See [19] for the proof.

Lemma 3.1.9

$\lim_{\varepsilon\to\infty} \|M_{AB}+\varepsilon\| - \varepsilon = \sup \ Re\, V(M_{AB})$.

Proof.

If f is a state, then for $\varepsilon > 0$ we have

$\|M_{AB}+\varepsilon\| \geq Re\, f(M_{AB}+\varepsilon) = Re\, f(\varepsilon) + Re\, f(M_{AB}) = \varepsilon + Re\, f(M_{AB})$.

$\Rightarrow \|M_{AB}+\varepsilon\| \geq \varepsilon + Re\, f(M_{AB})$

$\Rightarrow \|M_{AB}+\varepsilon\| - \varepsilon \geq Re\, f(M_{AB}) = \sup \ Re\, f(M_{AB}) = \sup \ Re\, V(M_{AB})$

Therefore, $\|M_{AB}+\varepsilon\| \geq \sup \ Re\, V(M_{AB})$.

On the other hand,

$$\begin{aligned}
\|M_{AB}+\varepsilon\| &= \|[(M_{AB})^2 - \varepsilon^2][M_{AB}-\varepsilon]^{-1}\| \\
&= \|(M_{AB})^2[M_{AB}-\varepsilon]^{-1} - \varepsilon^2[M_{AB}-\varepsilon]^{-1}\| \\
&\leq \varepsilon^2 d[\varepsilon, V(M_{AB})]^{-1} + o(1) \\
&\leq \varepsilon^2[\varepsilon - \sup \ Re\, V(M_{AB})]^{-1} + o(1).
\end{aligned}$$

Thus

$\limsup_{\varepsilon\to\infty} \|M_{AB}+\varepsilon\| - \varepsilon \leq \sup \ Re\, V(M_{AB})$. $\square$

Theorem 3.1.10

Let ω *be a complex number. Then* $\omega \in V(M_{AB})$ *if and only if*

$|\omega - \lambda| \leq \|M_{AB} - \lambda I_H\|$ *for all* $\lambda \in \mathbb{C}$. *Hence*

$V(M_{AB}) = \bigcap_{\lambda\in\mathbb{C}} \{z : |z-\lambda| \leq \|M_{AB} - \lambda I_H\|\}$.

Proof.

If $\omega = f(M_{AB}) \in V(M_{AB})$, then $|\omega - \lambda| = |f(M_{AB} - \lambda| \leq \|M_{AB} - \lambda I_H\|$ for any $\lambda \in \mathbb{C}$.

Conversely,

if ω is not an element of $V(M_{AB})$, then there is $\lambda \in \mathbb{C}$ such that

$|\omega - \lambda| > \|M_{AB} - \lambda I_H\|$.

By convexity of $V(M_{AB})$ we assume without loss of generality that $V(M_{AB})$ lies in the half plane with $Re\ z \leq 0$ and that $\omega > 0$. Thus by Lemma 3.1.9 we have that $\|M_{AB} + \varepsilon\| - \varepsilon < \omega$ for large positive ε, hence the proof. □

We recall that, norm of the basic elementary operator is defined by;

$$\begin{aligned}\|M_{AB}\| &= \sup\{M_{AB}(X) : \ X \in B(H), \ \|X\| = 1\} \\ &= \sup\{\|AXB\| : \ X \in B(H), \|X\| \leq 1\}.\end{aligned}$$

Theorem 3.1.11

Let $\mathcal{A}$ be C^*-algebra, then

$$\begin{aligned}\|M_{AB}\| &= \sup\{\|M_{AB}(U)\| : U \in U(\mathcal{A})\} \\ &= \sup\{\|AUB\| : \ U \in U(\mathcal{A})\}\end{aligned}$$

where $U(\mathcal{A})$ denotes the set of unitaries in $\mathcal{A}$.

For proof see [30].

Theorem 3.1.12

Let H be a complex Hilbert space and $B(H)$ the algebra of all bounded linear operators on H. Then, $V(M_{AB/B(B(H))}) = [\bigcup_{U \in U(B(H))} \overline{W(U^*AUB)}]$, for all $A, B \in B(H)$ and U a unitary operator.

Proof.

We first show that $[\bigcup_{U \in U(B(H))} \overline{W(U^*AUB)}] \subset V(M_{AB/B(B(H))})$

Let E be a Banach space. Then $T \in B(E)$ is said to be an isometry if $\|Tx\| = \|x\|$ for all $x \in E$. If T is an invertible isometry, then its inverse T^{-1} is also an isometry and therefore,

$$V(TST^{-1}{}_{/B(E)}) = V(S_{/B(E)}) \tag{3.1.6}$$

for all $S \in B(E)$.

If $E = H$ then $T = U$ and $T^{-1} = U^*$. Thus from equation (3.1.6) we have that

$$V(UAU^*{}_{/B(H)}) = V(U^*AU_{/B(H)}) = V(A_{/B(H)}) \tag{3.1.7}$$

for all $A \in B(H)$.

Given two unitaries $U, V \in B(H)$, then

$$V(M_{U^*AU \ \ V^*BV/B(B(H))}) = V(M_{AB/B(B(H))}) \tag{3.1.8}$$

for all $A \in B(H)$.

Now, taking an invertible isometry R_{UV^*} with R_{U^*V} as its inverse, then

$$V(M_{U^*AU \ V^*BV/B(B(H))}) = V(R_{UV^*} \ M_{AB} \ R_{U^*V/B(B(H))}),$$

and by lemma 3.1.5

$$W(U^*AU \ V^*BV) \subset V(R_{UV^*} \ M_{AB} \ R_{U^*V/B(B(H))})$$

and

$$\bigcup_{U,V \in U(B(H))} W(U^*AU \ V^*BV) \subset V(M_{AB/B(B(H))}).$$

Since the algebraic numerical range of the basic elementary operator is closed and the product of two unitaries is also a unitary, then

$$[\bigcup_{U \in U(B(H))} \overline{W(U^*AUB)}] \subset V(M_{AB/B(B(H))}) \tag{3.1.9}$$

or

$$[\bigcup_{V \in U(B(H))} \overline{W(V^*AVB)}] \subset V(M_{AB/B(B(H))}).$$

Next we proceed to show the inclusion
$V(M_{AB/B(B(H))}) \subset [\bigcup_{U \in U(B(H))} \overline{W(U^*AUB)}]$.

Now, if $\mathcal{A} = B(H)$ then $M_{AB}(U) = AUB$ for all $U \in U(B(H))$.

Therefore,

$V(M_{AB/B(B(H))}) = \bigcap_{z \in \mathbb{C}} \{\lambda : |\lambda - z| \leq \|M_{A,B} - zI_{B(H)}\|\}$.

But

$\|M_{AB} - z\| = \sup\{\|(M_{AB} - z)(U)\| : U \in U(B(H))\}$

$\qquad\qquad = \sup\{\|AUB - zU\| :\ U \in U(B(H))\}.$

Since the unitary $U \in U(B(H))$ is an isometry, then

$\|M_{AB} - zI_{B(H)}\| = \sup\{\|U^*AUB - zI_{B(H)}\| :\ U \in U(B(H))\}.$

So if $\mu \in V(M_{AB/B(B(H))})$ then for all $z \in \mathbb{C}$,

$\mu \in \{|\lambda - z| \leq \|M_{AB} - zI_{B(H)}\|\}.$

Taking a fixed $\varepsilon > 0$, there exists U_ε such that

$\|M_{AB} - zI_{B(H)}\| < \|U_\varepsilon^* AU_\varepsilon B - zI_{B(H)}\| + \varepsilon$ and by Theorem 3.1.10 we have that,

$$\begin{aligned} W(U_\varepsilon^* AU_\varepsilon B)^- &= V(U_\varepsilon^* AU_\varepsilon B) \\ &= \bigcap_{z\in\mathbb{C}} \{\lambda : |\lambda - z| \leq \|U_\varepsilon^* AU_\varepsilon B - zI_{B(H)}\|\} \end{aligned}$$

and so there exists $\lambda \in W(U_\varepsilon^* AUB)$ such that $|\mu - \lambda| \leq \varepsilon$.

Since ε is arbitrary, $\mu \in [\bigcup_{U\in U(B(H))} \overline{W(U^*AUB)}]$.

Thus

$$V(M_{AB/(B(BH))}) \subset [\bigcup_{U\in UB((H))} \overline{W(U^*AUB)}] \qquad (3.1.10).$$

Therefore, from equations 3.1.9 and 3.1.10 we have that

$V(M_{AB/(B(H))}) = [\bigcup_{U\in U(B(H))} \overline{W(U^*AUB)}]$. $\square$

Example 3.1.13

As an immediate calculation to theorem 3.1.12, we consider $B(H)$ to be $M_2(\mathbb{C})$ such that $M_2(\mathbb{C}) : \mathbb{C}^2 \to \mathbb{C}^2$.

We recall that by definition, $M_{AB} : B(H) \to B(H)$ is given by

$$M_{AB}(X) = AXB$$

for all $X \in B(H)$ with $\|X\| = 1$ and $A, B \in B(H)$ fixed.

Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ and $X = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Then we have that $AXB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

Now,

$$W\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \{\langle \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{matrix} x_1, & x_1 \\ x_2 & x_2 \end{matrix} \rangle : \|x\| \leq 1\}$$
$$= \{\langle \begin{matrix} x_1, & x_1 \\ 0 & x_2 \end{matrix} \rangle : |x_1|^2 + |x_2|^2 = 1\}$$
$$= \{x_1\overline{x_1} : |x_1|^2 + |x_2|^2 = 1$$
$$If x_1 = r_1 e^{i\theta} and x_2 = r_2 e^{i\theta} then$$
$$= \{x_1\overline{x_1} : r_1^2 + r_2^2 = 1\}$$
$$= \{|x_1|^2 : r_1^2 + r_2^2 = 1\}$$
$$= \{r_1^2 : r_1^2 + r_2^2 = 1\}$$
$$= \{1 - r_2^2 : r_1^2 + r_2^2 = 1\}$$
$$= \{1 - r_2^2 : 0 \leq r_2 \leq 1\}$$
$$= \{1 - x_2^2 : 0 \leq x_2 \leq 1\}.$$

So when $x_2 = 0$ we have that $|x_1|^2 = 1$ and $x_1 = 0$ gives $|x_1|^2 = 0$.
Therefore $W\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is a subset of [0,1] and contains 0,1.
Since the classical numerical range of an operator is convex, then the numerical range $W(AXB)$ is the closed interval $[0,1]$.
The closure of this interval $\overline{[0,1]}$ is again the closed interval $[0,1]$ and since $V(T) = \overline{W(T)}$, then we have that $V(M_{AB}(X))$ is the closed interval $[0,1]$.

Next we extend this result to the context of C*-algebra and show that
$V(M_{ab/B(\mathcal{A})}) = \bigcup\{V(u^*aub/\mathcal{A}) : u \in U(\mathcal{A})\}$, where $U(\mathcal{A})$ denotes the set of unitary elements. Here, the multiplication operator acts on a C*-algebra.
Let $\mathcal{A}$ be a unital C*-algebra. Then an element $u \in \mathcal{A}$ is called unitary if $u^*u = uu^* = 1$, that is, u is invertible and $u^* = u^{-1}$. Also,
$V(u^{-1}au/\mathcal{A}) = V(a/\mathcal{A})$, for any $a,\ u \in \mathcal{A}$.

Proposition 3.1.14

Let $\mathcal{A}$ be a C-algebra with $a,\ b \in \mathcal{A}$. Then*
$V(M_{ab/B(\mathcal{A})}) = \bigcup\{V(u^*aub/\mathcal{A}) : u \in U(\mathcal{A})\}$, *where $U(\mathcal{A})$ denotes the set of unitary elements.*

Proof.

We begin by showing that;

$\bigcup\{V(u^*aub/\mathcal{A})\ u \in U(A)\} \subset V(M_{ab/B(\mathcal{A})})$.

The norm of M_{ab} is given by;

$$\begin{aligned}\|M_{ab}\| &= \sup\{\|M_{ab}(x)\| : \|x\| \leq 1\} \\ &= \sup\{\|axb\| :\ x \in_A,\ \|x\| \leq 1\}\end{aligned}$$

and by Theorem 3.1.5,

$V(M_{ab/B(\mathcal{A})}) = \bigcap_{z\in\mathbb{C}}\{\lambda : |\lambda - z| \leq \|M_{ab} - z\|\}$

But,

$$\begin{aligned}\|M_{ab} - z\| &= \sup\{\|(M_{ab} - z)(u)\| : \|u\| \leq 1\} \\ &= \sup\{\|(aub - zu)\| : \|u\| \leq 1\} \\ &= \sup\{\|u^*aub - z\| : \|u\| \leq 1\}\end{aligned}$$

for all $u \in U(\mathcal{A})$.

So,

$$\begin{aligned}V(u^{-1}aub/\mathcal{A}) &= \bigcap_{z\in\mathbb{C}}\{\lambda : |\lambda - z| \leq \|u^{-1}aub - z\|\} \\ &= \bigcap_{z\in\mathbb{C}}\{\lambda : |\lambda - z| \leq \|aub - zu\|\} \\ &= \bigcap_{z\in\mathbb{C}}\{\lambda : |\lambda - z| \leq \|(M_{ab} - z)\|\} \\ &= V(M_{ab}).\end{aligned}$$

Since

$\|(M_{ab} - z)(u)\| \leq \|M_{ab} - z\|$ and

$\|M_{ab}(u)\| \leq \|M_{ab}\|$, then

$\{V(u^{-1}aub/\mathcal{A}) : u \in U(\mathcal{A})\} \subset V(M_{ab}/B(\mathcal{A}))$. Since $u^{-1} = u^*$ and by [21] Russo-Dye's theorem that a closed unit ball in $\mathcal{A}$ is the closed convex hull of $U(\mathcal{A})$ for a unital C*-algebra $\mathcal{A}$ and a unitary group U, then

$\bigcup\{V(u^*aub/\mathcal{A}) : u \in U(\mathcal{A})\} \subset V(M_{ab/B(\mathcal{A})})$.

Next we shall prove the inlusion

$V(M_{ab/B(\mathcal{A})}) \subset \bigcup\{V(u^*aub/\mathcal{A}) : u \in U(\mathcal{A})\}$.

From Theorem 3.1.6,
$V(M_{ab/B(\mathcal{A})}) = \bigcap_{z\in\mathbb{C}}\{\lambda : |\lambda - z| \leq \|M_{ab} - z\|\}$
But

$$\begin{aligned}\|M_{ab} - z\| &= \sup\{\|(M_{ab} - z)(u)\| : u \in U(\mathcal{A}), \|u\| = 1\}\\ &= \sup\{\|(aub - zu)\| : u \in U(\mathcal{A}), \|u\| = 1\}\\ &= \sup\{\|u^*aub - zu\| : u \in U(\mathcal{A}), \|u\| = 1\}.\end{aligned}$$

So if $\alpha \in V(M_{ab/B(\mathcal{A})})$, then for all $z \in \mathbb{C}$, $\alpha \in \{|\lambda - z| \leq \|M_{ab} - z\|\}$.
Taking a fixed $\varepsilon > 0$, then there exists a unitary $u_\varepsilon \in \mathcal{A}$ such that
$\|M_{ab} - z\| < \|u_\varepsilon^* a u_\varepsilon b - zu\| + \varepsilon$.
But

$$\begin{aligned}(u_\varepsilon^* a u_\varepsilon b/\mathcal{A}) &= \bigcap_{z\in\mathbb{C}}\{\lambda : |\lambda - z| \leq \|u_\varepsilon^* a u_\varepsilon b - z\|\}\\ &= \bigcap_{z\in\mathbb{C}}\{\lambda : |\lambda - z| \leq \|a u_\varepsilon b - z u_\varepsilon\|\}.\end{aligned}$$

Hence there exists $\alpha \in V(u_\varepsilon^* a u_\varepsilon b/\mathcal{A})$ such that $|\alpha - \lambda| \leq \varepsilon$ and since ε is arbitrary, then $\alpha \in \bigcup_{u\in U(\mathcal{A})} V(u^*aub/\mathcal{A})$. □

3.2 Norm of a generalized derivation

In this section we determine the norm of a generalized derivation. Let H be a separable infinite dimensional complex Hilbert space and let $B(H)$ denote the algebra of all bounded linear operators on H. Let $A, B \in B(H)$. The left and the right multiplication operators induced by A and B is denoted by L_A and R_B respectively and defined by

$$L_A(X) = AX$$

and

$$R_B(X) = XB.$$

The generalized derivation $\delta_{AB} : B(H) \to B(H)$ is defined by

$$\delta_{AB}(X) = (L_A - R_B)(X) = AX - XB. \qquad (3.2.1)$$

for all $X \in B(H)$

Proposition 3.2.1

The generalized derivation $\delta_{AB} : B(H) \to B(H)$ is a linear bounded map.

Proof.

We first show that the generalized derivation is linear.

Let $\mu, \omega \in \mathbb{C}$ and fix $A, B \in B(H)$. Then for all $S, T \in B(H)$, we have that $\mu S + \omega T \in B(H)$. So

$$\begin{aligned}\delta_{AB}(\mu S + \omega T) &= A(\mu S + \omega T) - (\mu S + \omega T)B \\ &= \mu AS + \omega AT - \mu SB - \omega TB \\ &= (\mu AS - \mu SB) + (\omega AT - \omega TB) \\ &= \mu(AS - ST) + \omega(AT - TB) \\ &= \mu\delta_{AB}(S) + \omega\delta_{AB}(T).\end{aligned}$$

Next, we show that δ_{AB} is bounded on $B(H)$.

Now, for all $X \in B(H)$ with $\|X\| = 1$ and A, B fixed in $B(H)$, we have that

$$\begin{aligned}\|\delta_{AB}(X)\| &= \|AX - XB\| \\ &\leq \|AX\| + \|XB\| \\ &\leq \|A\|\|X\| + \|X\|\|B\| \\ &= \|A\| + \|B\|.\square\end{aligned}$$

The following result which show equality between $\|\delta_{AB}(X)\|$ and $\|A\| + \|B\|$ is proved for only finite rank operators in $B(H)$.

Note that for any space $B(H)$, the set of all finite rank operators on H is a subspace of $B(H)$. We denote this set by $\mathcal{F}(H)$.

Theorem 3.2.2

Let $B(H)$ be the set of all linear bounded operators from H to H and assume that $\mathcal{F}(H) \subset B(H)$ be the subspace of finite rank operators from H to H. Then $\|\delta_{AB/\mathcal{F}(H)}\| = \|A\| + \|B\|$ for all $A, B \in \mathcal{F}(H)$.

Proof.

By definition,

$$\begin{aligned}\|\delta_{AB/\mathcal{F}(H)}\| &= \sup\{\|\delta_{AB}(X)\| : X \in \mathcal{F}(H),\ \|X\| = 1\} \\ &= \sup\{\|AX - XB\| : X \in \mathcal{F}(H), \|X\| = 1\}.\end{aligned}$$

Therefore,

$\|\delta_{AB/\mathcal{F}(H)}\| \geq \|\delta_{AB}(X)\|$ for all $X \in \mathcal{F}(H)$ and $\|X\| = 1$.

Taking an arbitrary $\varepsilon > 0$ we have

$\|\delta_{AB/\mathcal{F}(H)}\| - \varepsilon < \|\delta_{AB}(X)\|$ for all $X \in \mathcal{F}(H)$ and $\|X\| = 1$. So

$\|\delta_{AB/\mathcal{F}(H)}\| - \varepsilon < \|AX - XB\|$.

Since$\|AX - XB\| \leq \|A\| + \|B\|$ and letting $\varepsilon \to 0$, then we have that

$$\|\delta_{AB/\mathcal{F}(H)}\| \leq \|A\| + \|B\| \qquad (3.2.2).$$

On the other hand,

Let $s, y, z \in H$ be unit vectors and let u, v be functionals so that $u \otimes y : H \to \mathbb{C}$ and $v \otimes z : H \to \mathbb{C}$ are finite rank operators defined by

$$(u \otimes y)s = u(s)y$$

and

$$(v \otimes z)s = v(s)z$$

for all $s \in H$ with $\|s\| = 1$.

So

$$\begin{aligned}\|u \otimes y\| &= \sup\{\|(u \otimes y)s\| : s \in H,\ \|s\| = 1\} \\ &= \sup\{\|u(s)y\| : s \in H,\ \|s\| = 1\} \\ &= \sup\{|u(s)|\|y\| : s \in H,\ \|s\| = 1\} \\ &= \|u\|.\end{aligned}$$

Similarly, $\|v \otimes z\| = \|v\|$.

So if we let $A = u \otimes y$ and $B = v \otimes z$, then $\|A\| = |u(s)| = \|u\|$ and $\|B\| = |v(s)| = \|v\|$.

Now,
$\|\delta_{AB/\mathcal{F}(H)}\| \geq \|\delta_{AB}(X)\| \geq \|\delta_{AB}(X)s\|$ where $X \in \mathcal{F}(H)$ with $\|X\| = 1$.
But,

$$\begin{aligned}\delta_{AB}(X)s &= (AX - XB)(s)\\ &= AX(s) - XB(s)\\ &= ((u \otimes y)X(s)) - (X(v \otimes z))(s)\\ &= u(s)yX - Xv(s)z\\ &= u(s)X(y) - X(z)v(s.)\end{aligned}$$

Therefore,
$\|\delta_{AB/\mathcal{F}(H)}\|^2 \geq \|(AX - XB)(s)\|^2$
But,

$$\begin{aligned}\|(AX - XB)(s)\|^2 &= \langle u(s)X(y) - X(z)v(s), u(s)X(y) - X(z)v(s)\rangle\\ &= \langle u(s)X(y), u(s)X(y)\rangle - \langle u(s)X(y), X(z)v(s)\rangle - \langle X(z)v(s), u(s)X(y)\rangle\\ &+ \langle X(z)v(s), X(z)v(s)\rangle\\ &= \|u(s)X(y)\|^2 - \langle u(s)X(y), X(z)v(s)\rangle - \langle X(z)v(s), u(s)X(y)\rangle + \|X(z)v(s)\|^2\\ &= |u(s)|^2\|X(y)\|^2 - (uX(y)X(z)v)\langle s, s\rangle - (X(z)vuX(y))\langle s, s\rangle + \|X(z)\|^2|v(s)|^2\\ &= |u(s)|^2 - uX(y)vX(z) - vX(z)uX(y) + |v(s)|^2\\ &= \|u\|^2 - uX(y)vX(z) - vX(z)uX(y) + \|v\|^2.\end{aligned}$$

Setting $uX(y) = |uX(y)| = \|A\|$, and
$vX(z) = -|vX(z)| = -\|B\|$ then we have that

$$\begin{aligned}\|u\|^2 - uX(y)vX(z) - vX(z)uX(y) + \|v\|^2 &= \|A\|^2 + 2\|A\|\|B\| + \|B\|^2\\ &= (\|A\| + \|B\|)^2.\end{aligned}$$

Thus,
$\|\delta_{AB/\mathcal{F}(H)}\|^2 \geq (\|A\| + \|B\|)^2$.
Taking square root on both sides we obtain

$$\|\delta_{AB/\mathcal{F}(H)}\| \geq \|A\| + \|B\| \qquad (3.2.3).$$

Equations (3.2.2) and (3.2.3) together yields,
$\|\delta_{AB/\mathcal{F}(H)}\| = \|A\| + \|B\|$. □

The result in Theorem 3.2.2 can only be extended to the whole of $B(H)$ with certain conditions on the numerical ranges $W_o(A)$ and $W_o(B)$. This is done using Stampfli's maximal numerical range.

We recall that for $A \in B(H)$ the maximal numerical range of A is given by

$$W_o(A) = \{\lambda \in \mathbb{C} : \langle Ax_n, x_n\rangle \to \lambda, \ with \ \|x_n\| = 1 \ and \ \|Ax_n\| \to \|A\|\}.$$

Lemma 3.2.3 and Theorem 3.2.4 give detailed conditions on $W_o(A)$ and $W_o(B)$ for the results of Theorem 3.2.2 to hold in case of $B(H)$.

Lemma 3.2.3
Let $\lambda_1 \in W_o(A)$ *and* $\lambda_2 \in W_o(B)$. *Then*
$\|\delta_{AB/B(H)}\| \geq (\|A\|^2 - |\lambda_1|^2)^{\frac{1}{2}} + (\|B\|^2 - |\lambda_2|^2)^{\frac{1}{2}}$.

Proof.
By definition,

$$\|\delta_{AB}/B(H)\| = \sup\{\|AX - XB\| : X \in B(H) \ and \ \|X\| = 1\}.$$

Since $\lambda_1 \in W_o(A)$, there exists $x_n \in H$ such that $\|Ax_n\| \to \|A\|$ and $\langle Ax_n, x_n\rangle \to \lambda_1$. Also, for $\lambda_2 \in W_o(B)$, there exists $x_n \in H$ such that $\|Bx_n\| \to \|B\|$ and $\langle Bx_n, x_n\rangle \to \lambda_2$.
We set $Ax_n = \alpha_n x_n + \beta_n y_n$ and $Bx_n = \alpha_n x_n + \omega_n y_n$ where $\langle x_n, y_n\rangle = 0$, $\|y_n\| = 1$. Also, let β_n and ω_n be either both positive or both negative.

Given that $V_n x_n = x_n$, $V_n y_n = -y_n$ and $V_n = 0$ on $\{x_n, y_n\}^{\perp}$, then

$$\begin{aligned}\|(AV_n - V_nB)x_n\| &= \|AVx_n - V_nBx_n\| \\ &= \|Ax_n - V_n(\alpha_n x_n + \omega_n y_n)\| \\ &= \|Ax_n - V_n\alpha_n x_n - V_n\omega_n y_n\| \\ &= \|\alpha_n x_n + \beta_n y_n - \alpha_n x_n + \omega_n y_n\| \\ &= \|\beta_n y_n + \omega_n y_n\| \\ &= |\beta_n + \omega_n| \\ &= |\beta_n| + |\omega_n|.\end{aligned}$$

But

$\|Ax_n\| = \|\alpha_n x_n + \beta_n y_n\| \leq \|\alpha_n x_n\| + \|\beta_n y_n\| = |\alpha_n| + |\beta_n|$.

So $|\beta_n| \geq \|Ax_n\| - |\alpha_n|$ and since $\|Ax_n\| \to \|A\|$, then

$|\beta_n| \geq (\|A\|^2 - |\alpha_n|^2)^{\frac{1}{2}} - \varepsilon_n$ where $\varepsilon_n \to 0$ and $\alpha_n \to \lambda_1$.

Also,

$\|Bx_n\| = \|\alpha_n x_n + \omega_n y_n\| \leq \|\alpha_n x_n\| + \|\omega_n y_n\| = |\alpha_n| + |\omega_n|$

So $|\omega_n| \geq \|Bx_n\| - |\alpha_n|$ and since $\|Bx_n\| \to \|B\|$ then

$|\omega_n| \geq (\|B\|^2 - |\alpha_n|^2)^{\frac{1}{2}} - \varepsilon_n$ where $\varepsilon_n \to 0$ and $\alpha_n \to \lambda_2$

Thus

$$\begin{aligned}|\beta_n + \omega_n| = |\beta_n| + |\omega_n| &\geq (\|A\|^2 - |\alpha_n|^2)^{\frac{1}{2}} - \varepsilon_n + (\|B\|^2 - |\alpha_n|^2)^{\frac{1}{2}} - \varepsilon_n \\ &= (\|A\|^2 - |\lambda_1|^2)^{\frac{1}{2}} + (\|B\|^2 - |\lambda_2|^2)^{\frac{1}{2}}.\end{aligned}$$

Therefore,

$\|\delta_{AB/B(H)}\| \geq \|\delta_{AB}(V_n)\| \geq \|(AV_n - V_nB)x_n\| \geq (\|A\|^2 - |\lambda_1|^2)^{\frac{1}{2}} + (\|B\|^2 - |\lambda_2|^2)^{\frac{1}{2}}$.

Similarly,

if λ_1 and λ_2 are as defined in lemma 3.2.3 and we let $\alpha_n = \langle Ax_n, x_n\rangle \to \lambda_1$ and $\alpha_n = \langle Bx_n, x_n\rangle \to \lambda_2$ so that

$|\alpha_n|^2 + |\beta_n|^2 = \|Ax_n\|^2 \to \|A\|^2$ that is, $|\beta_n| = (\|AX_n\|^2 - |\alpha_n|^2)^{\frac{1}{2}}$ and

$|\alpha_n|^2 + |\omega_n|^2 = \|Bx_n\|^2 \to \|B\|^2$ that is, $|\omega_n| = (\|Bx_n\|^2 - |\alpha_n|^2)^{\frac{1}{2}}$.

Also, let $V_n = x_n \otimes x_n - y_n \otimes y_n$, then $\|V_n\| = 1$ and $(AV_n - V_nB)x_n = \beta_n y_n + \omega_n y_n$.

Thus

$$\begin{aligned}\|\delta_{AB/B(H)}\| \geq \|(AV_n - V_nB)x_n\| = \|\beta_n y_n + \omega_n y_n\| &= |\beta_n| + |\omega_n| \\ &= (\|Ax_n\|^2 - |\alpha_n|^2)^{\frac{1}{2}} + (\|Bx_n\|^2 - |\alpha_n|^2)^{\frac{1}{2}}\end{aligned}$$

$$= (\|Ax_n\|^2 - |\langle Ax_n, x_n\rangle|^2)^{\frac{1}{2}} + (\|Bx_n\|^2 - |\langle Bx_n, x_n\rangle|^2)^{\frac{1}{2}}$$
$$\to (\|A\|^2 - |\lambda_1|^2)^{\frac{1}{2}} + (\|B\|^2 - |\lambda_2|^2)^{\frac{1}{2}}. \ \square \qquad (3.2.4)$$

Now, by application of results in Lemma 3.2.3, we prove that equality exists between $\|\delta_{AB/B(H)}\|$ and $\|A\| + \|B\|$ if and only if $W_o(A) \bigcap W_o(B)$ at least contains zero.

Theorem 3.2.4

Let $\delta_{AB} : B(H) \to B(H)$ *be the generalized derivation and assume that* $0 \in W_o(A) \bigcap W_o(B)$. *Then* $\|\delta_{AB/B(H)}\| = \|A\| + \|B\|$ *if and only if* $0 \in W_o(A) \bigcap W_o(B)$.

Proof.

Let $0 \in W_o(A)$ and $0 \in W_o(B)$, then by lemma 3.2.3 we have that

$\|\delta_{AB/B(H)}\| \geq \|A\| + \|B\|$.

Since $\|\delta_{AB/B(H)}\| \leq \|A\| + \|B\|$ for fixed $A, B \in B(H)$ then

$\|\delta_{AB/B(H)}\| = \|A\| + \|B\|$.

Conversely,

let $\|\delta_{AB/B(H)}\| = \|A\| + \|B\|$.

We show that if $\langle Ax_n, x_n\rangle \to \lambda_1$ and $\langle Bx_n, x_n\rangle \to \lambda_2$ then $\lambda_1, -\lambda_1 \in W_o(A)$ and $\lambda_2, -\lambda_2 \in W_o(B)$

Since $\|\delta_{AB/B(H)}\| = \|A\| + \|B\|$, there exists $x_n, V_n \in B(H)$ such that

$\|x_n\| = \|V_n\| = 1$ and $\|(AV_n - V_nB)x_n\| \to \|A\| + \|B\|$ so that

$\|V_nx_n\| = 1$, $\|Ax_n\| \to \|A\|$, $\|Bx_n\| \to \|B\|$, $\|AV_nx_n\| \to \|A\|$ and $\|BV_nx_n\| \to \|B\|$.

Moreover since $\|(AV_n - V_nB)x_n\| \to \|A\| + \|B\|$, then

$AV_nx_n = -V_nAx_n + \overrightarrow{\varepsilon_n}$ where $\|\overrightarrow{\varepsilon_n}\| \to 0$ and

$V_nBx_n = -V_nBx_n + \overrightarrow{\varepsilon_n}$ where $\|\overrightarrow{\varepsilon_n}\| \to 0$

Let $\langle Ax_n, x_n\rangle \to \lambda_1$ and $\langle Bx_n, x_n\rangle \to \lambda_2$ that is $\lambda_1 \in W_o(A)$ and $\lambda_2 \in W_o(B)$.

By choosing a subsequence we have that

$$\begin{aligned}\langle AV_nx_n, V_nx_n\rangle &= -\langle V_nAx_{n,V_nx_n}\rangle \overrightarrow{\varepsilon_n} \\ &= -\langle Ax_n, V_n^*V_nx_n\rangle + \overrightarrow{\varepsilon_n} \\ &= -\langle Ax_n, x_n\rangle + \overrightarrow{\varepsilon_n}.\end{aligned}$$

Thus

$\lim_{n\to\infty}\langle AV_nx_n, V_nx_n\rangle \to -\lambda_1$ that is, $-\lambda_1 \in W_o(A)$.
Therefore $0 \in W_o(A)$.
Similarly,

$$\begin{aligned}\langle V_nBx_n, V_nx_n\rangle &= -\langle V_nBx_n, V_nx_n\rangle + \overrightarrow{\varepsilon_n}\\ &= -\langle Bx_n, V_n^*V_nx_n\rangle + \overrightarrow{\varepsilon_n}\\ &= -\langle B_nx_n, x_n\rangle + \overrightarrow{\varepsilon_n}.\end{aligned}$$

Thus
$\lim_{n\to\infty}\langle V_nBx_n, V_nx_n\rangle = -\lambda_2$ that is, $-\lambda_2 \in W_o(B)$.
Therefore, $0 \in W_o(B)$. $\square$

The above results can always be extended to the space $M_n(\mathbb{C})$, the space of all $n \times n$ matrices, with finite entries via the Gelfand-Naimark Segal theorem since by construction there is always an isomorphism representation π between $B(H)$ and $M_n(\mathbb{C})$ implying that the two spaces are isomorphic.
Thus, $\|\delta_{AB/M_n(\mathbb{C})}\| = \|A\| + \|B\|$ if and only if $0 \in W_o(A) \bigcap W_o(B)$ where $A, B \in M_n(\mathbb{C})$.
A C*-algebra $\mathcal{A}$ is said to be irreducible if the commutant of $\mathcal{A}$ contains only the scalars.
As mentioned earlier, note that $B(H)$ is isomorphic to C*-algebras of the form $M_n(\mathbb{C})$.

Let $A \in B(H)$, $B(H)$ an irreducible C*-algebra and define the distance of A from a scalar multiple of the appropriate identity operator $I \in B(H)$ denoted as $d(A)$ by

$$d(A) = \inf\{\|A - \lambda\| : \lambda \in \mathbb{C}\}. \tag{3.2.5}$$

Theorem 3.2.5
Let $\delta_{AB} : B(H) \to B(H)$ be a generalized derivation, $B(H)$ an irreducible C-algebra and suppose $d(A)$ and $d(B)$ are distances from A and B respectively as defined in (3.2.5), then $\|\delta_{AB/B(H)}\| = d(A) + d(B)$.*

Proof.

Following Stampfli's proof, $\|\delta_{AB/B(H)}\| = \inf_{\lambda\in\mathbb{C}}(\|A-\lambda\| + \|B-\lambda\|)$.

Since $d(A) = \inf_{\lambda\in\mathbb{C}} \|A-\lambda\|$ and $d(B) = \inf_{\lambda\in\mathbb{C}} \|B-\lambda\|$, we need to prove that for all $A, B \in B(H)$

$$d(A) + d(B) = \inf_{\lambda\in\mathbb{C}}(\|A-\lambda\| + \|B-\lambda\|).$$

It therefore suffices to show that if $\mathcal{X}$ and $\mathcal{Y}$ are bounded sets, then

$$\inf(\mathcal{X}+\mathcal{Y}) = \inf\mathcal{X} + \inf\mathcal{Y}.$$

Now,

$\inf\mathcal{X} \leq x$ and $\inf\mathcal{Y} \leq y$ for all $x \in \mathcal{X}$ and $y \in \mathcal{Y}$ implies that

$\inf\mathcal{X} + \inf\mathcal{Y} \leq x + y$ for all $x \in \mathcal{X}$ and $y \in \mathcal{Y}$

$\Rightarrow \inf\mathcal{X} + \inf\mathcal{Y} \leq \inf(\mathcal{X}+\mathcal{Y})$.

On the other hand,

Let $\varepsilon > 0$ be given.

Then by definition of infimum, there exists $x \in \mathcal{X}$ and $y \in \mathcal{Y}$ such that

$x < \inf\mathcal{X} + \frac{\varepsilon}{2}$ and

$y < \inf\mathcal{Y} + \frac{\varepsilon}{2}$

$\Rightarrow x + y < \inf\mathcal{X} + \inf\mathcal{Y} + \varepsilon$

Since ε is arbitrary, we have that

$x + y < \inf\mathcal{X} + \inf\mathcal{Y}$ which completes the proof. $\square$

CHAPTER FOUR

CONCLUSION AND RECOMMENDATIONS

In this chapter we conclude and provide some recommendations for further research.

4.1 Conclusion

In this thesis, in proposition 3.1.1, we have shown that some basic properties of the numerical range of an operator $T \in B(H)$ holds for the algebraic numerical range of the basic elementary operator. The convexity of the algebraic numerical range of the basic elementary operator has been proved in theorem 3.1.2 and that the algebraic numerical range of the basic elementary operator contains the closure of its classical numerical range in theorem 3.1.3. Using lemma 3.1.4, theorem 3.1.5, lemma 3.1.6, theorem 3.1.7, lemma 3.1.9 and theorem 3.1.10, we have proved that the algebraic numerical range of the basic elementary operator is equal to the closure of classical numerical range of the operators that implement it in theorem 3.1.11. Further, in proposition 3.1.14 we have extended the result in theorem 3.1.11 to the context of C*-algebra and proved that the algebraic numerical range of the basic elementary operator is equal to the algebraic numerical range of its implementing operators.

For the generalized derivation, we have shown in proposition 3.2.1 that it is linear and bounded. We have used finite rank operators in theorem 3.2.2 to prove that the norm of a generalized inner derivation has also been shown to be equal to the sum of the norms of the operators that induce it. Further, using Stampfli's results on maximal numerical range in lemma 3.2.3 and theorem 3.2.4, we have proved that the norm equality holds. Finally we have shown that $\|\delta_{AB/B(H)} = d(A) + d(B)$ in theorem 3.2.5.

4.2 Recommendation

From the results obtained from this study, we recommend the following for further research.

1. In this study we have determined the algebraic numerical range of the basic elementary operator. Further investigations should be done using other generalization of numerical range such as joint numerical range and even extended to other forms of elementary operators such as the Jordan elementary operator.
2. We have established the norm of a generalized derivation using finite rank operators and Stampfli's maximal numerical range. This can still be investigated under some conditions or special classes of operators.

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