ANALYSIS OF THE GREEK PARAMETERS OF A NONLINEAR BLACK-SCHOLES PARTIAL DIFFERENTIAL EQUATION

by

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Abstract

Derivatives are used in hedging European options against risks. The partial derivatives of the solution with respect to either a variable or a parameter in the Black-Scholes model are called risk parameters or simply the Greeks. Nonlinear versions of the standard Black-Scholes Partial Differential Equation have been introduced in financial mathematics in order to deal with illiquid markets. Market liquidity is relevant in the risk management of derivatives since in an illiquid market the implementation of a dynamic hedging strategy affects the price process of the underlying. Different hedging strategies and suitable pricing adjustments are needed. We studied the Greek parameters of a nonlinear Black-Scholes Partial Differential Equation whose nonlinearity is as a result of transaction costs for modelling illiquid markets. The objective of this study was to compute the Greek parameters of a European call option in illiquid markets whose illiquidity is arising from transaction costs. This is in relation to Cetin *et al.* model in which transaction costs have been incorporated (with zero interest rate). These Greeks were compared with those derived from the formula of Bakstein and Howison (2003) equation (with positive interest rate). All these Greeks were of the form $a + \frac{1}{\rho}f(S,t)$. The methodology involved deriving the Greek parameters from the formula of the equation by differentiating the formula with respect to either a variable or a parameter. These Greeks may help a trader to hedge risks in a non-ideal market situation. Greeks show how to protect one's position against adverse movements in critical market variables such as the stock price, time and interest rate.

CHAPTER 1

Introduction

This thesis is organized as follows: Chapter 1 is an introductory chapter. Basic concepts are also discussed in this chapter. Chapter 2 provides a brief history on stock price modelling for both linear and nonlinear Black-Scholes equation. The next chapter deals with the theory of the linear Black-Scholes valuation model. Finally, Chapter 4 focuses on the results and discussion of the Greek parameters of the nonlinear Black-Scholes equation. Thereafter some concluding remarks and ideas for future research are given.

1.1 Background Information

The famous Black and Scholes equation provides markets with a way of pricing options. The derivation of this equation is based on the assumption that markets are complete, frictionless and perfectly liquid. In a frictionless market, there are no transaction costs and restrictions on trade. In perfectly liquid markets investors can trade large volumes of stock without affecting their prices.

Risk management is concerned with controlling three financial risks that is market risk, credit risk and liquidity risk. In risk management, financial models based on the assumptions above may fail when the market faces poor liquidity. Investors are exposed to transaction costs in the form of commissions, fees and bid-ask-spread. Investors incur costs due to illiquidity if they choose to trade options during a period of poor liquidity. Purchase and sell of the underlying when there are costs incurred in trading the asset leads to market illiquidity. In illiquid markets the attempt to trade at a given point in time moves prices against the trader. A number of models for studying the pricing and the hedging of securities in illiquid markets or in the presence of transaction costs have been developed. In many of these models, derivative prices are characterized by fully nonlinear versions of the standard parabolic Black-Scholes equation.

Due to the introduction of liquidity risk in the market, the Greek parameters derived from the Black-Scholes formulae in the classical theory become unrealistic. The Greeks resulting from the Black-Scholes formula for valuing options in illiquid markets therefore, are appropriate in explaining this liquidity risk. These Greek parameters were obtained by differentiating the formula with respect to either a variable or a parameter. These derivatives are important in the hedging of an option position hence playing key roles in risk management. One of these models is the transaction cost model put forward by Cetin *et al.* [9]. This model takes into account the illiquidities arising from transaction costs. The purpose of this research was to compute and analyze the Greek parameters from the Black-Scholes formula of a non-linear equation and apply the stock data from NSE to these Greeks to see if they are applicable in a real life situation. We considered the European call options only since the formula from which we used to derive the Greek parameters was for the European call option.

The aim of this thesis was to compute the Greek parameters of a European call option in illiquid markets whose illiquidity is arising from transaction costs. This is in relation to Cetin *et al.* model in which transaction costs have been incorporated. These Greeks were compared with those derived from the formula of Bakstein and Howison (2003) equation. Recent studies have focussed on the Greek parameters in illiquid markets. In this study we mention the contributions of Esekon [12, 13, 14, 15].

1.2 Statement of the Problem

The linear Black-Scholes equation has been derived under the assumptions of complete, frictionless and perfectly liquid markets. Relaxing these assumptions brings in the issue of liquidity risk. This means that the Greek parameters derived from the Black-Scholes formula of a European call option under the classical theory cannot explain this illiquidity in the market. The Greek parameters arising from the nonlinear equation when $r > 0$ have been computed in Esekon [12] while those with $r = 0$ for a particular solution of a nonlinear equation have not been computed. There is need to compute the Greek parameters in cases when $r = 0$ if we are to compare these Greeks to those when $r > 0$.

1.3 Objectives of the Study

i) To compute the Greek parameters such as *delta, gamma, theta, speed* and *vega* of a European call option in illiquid markets whose illiquidity is arising from transaction costs.

ii) To compare Greeks in cases when $r = 0$ and $r > 0$.

1.4 Research Methodology

In this study we focused on European call options only. We derived the Greek parameters for a European call option from the Black-Scholes formula of a nonlinear Black-Scholes equation by differentiating it with respect to either a variable or a parameter in the formula derived in [13] (see Theorem 3.0.2, Equation (3.9)). We used a subset of stock price data from the Nairobi Securities Exchange to test the applicability of these parameters in a real life situation. A comparison of these Greeks was made with their counterparts in the nonlinear Black-Scholes formula for $r > 0$ in [12].

1.5 Significance of the Study

The Greeks obtained may help market makers to understand the liquidity risk in a non-ideal market situation. Greeks show how to protect one's position against adverse movements in critical market variables such as the stock price, volatility, time and interest rate. Also,financial institutions which trade options can manage their risk by use of these Greeks.

1.6 Basic Concepts

Brownian Motion

The arithmetic Brownian motion is given by

$$dS_t = \mu dt + \sigma dW_t, \tag{1.1}$$

where dS_t is the change in stock price S_t at time t, μ is the drift (the expected rate of return), σ is the volatility of the stock price and W_t is the Wiener process. Brownian

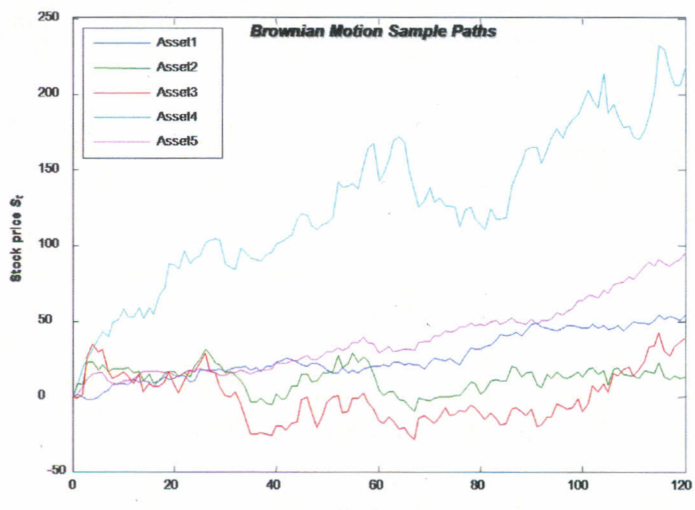

Figure 1.61: Brownian motion

motion, which was originally used as a model for stock price movements in 1900 by L. Bachelier [1] is a stochastic process $\{W_t : 0 \leq t \leq \infty\}$ characterized by the following properties:

(i) $W_0 = 0$,

(ii) Brownian motion $W_t \sim N(0, t)$. The increment $W_t - W_s \sim N(0, t-s)$ for $(0 \leq s < t)$,

(iii) W_t has stationary and independent increments i.e. for any positive integer n and any $0 \leq t_0 < t_1 < \cdots < t_n$ the random variables $W_{t_i} - W_{t_{i-1}}$is monotonic, $i = 1, \cdots, n$

are mutually independent and $W_{s+t} - W_s$ have the same distribution as W_t for any $s, t > 0$,

(iv) Brownian motion is a Markov process i.e. the distribution of its future values, conditional on the past and the present values, depends only on the present value and not on the past,

(v) Brownian motion is a martingale: $E[W_{t+1}|W_t, W_{t-1}, \cdots, W_1] = W_t$. This means that the expected value of a future outcome of Brownian motion conditional on the past and present information is exactly equal to the present value,

(vi) Brownian motion is continuous everywhere and differentiable nowhere.

The standard Brownian motion process has a drift rate of zero and a variance of one. We write this as

$$S_t = \mu t + \sigma W_t, \tag{1.2}$$

where $W_t = \varepsilon\sqrt{t}$ and ε is the standard Brownian motion i.e. $\varepsilon \sim N(0, 1)$. During the time step dt we get the SDE (1.1) from equation (1.2). The major weakness of this model was that the asset price could take negative value which is not realistic (see Figure 1.61).

Geometric Brownian Motion

In the case of Brownian motion process, a constant drift rate was assumed which is not the case of stock prices. For stock prices the return on investment is assumed to be constant where the rate of return at a given time is the ratio of the drift rate to the value of the stock at that time.

Hence the constant expected drift rate assumption in the case of Brownian motion process is inappropriate and needs to be replaced by an assumption of constant expected rate of return. Therefore a better model for stock price behaviour over time is given by

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t \tag{1.3}$$

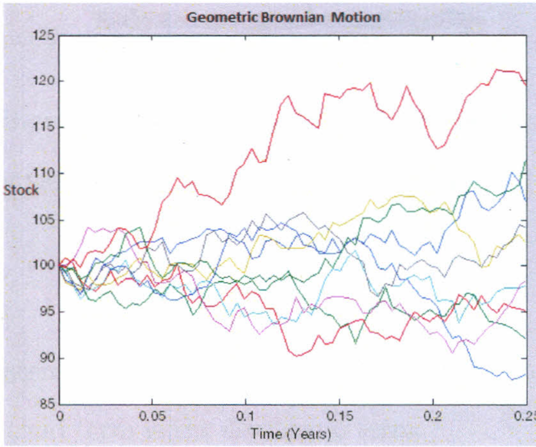

Figure 1.62: Geometric Brownian motion

Equation (1.3) is called a *generalized Wiener process.* The process assumes that the stock price returns $\frac{dS_t}{S_t}$ follow a stochastic process. Since S_t is the stock price at time t, then $\frac{dS_t}{S_t}$ is the rate of return on the asset. To get S_t we integrate equation (1.3) with respect to t to get

$$\ln \frac{S_t}{S_0} = \mu t + \sigma \int_0^t dW_t$$

which gives the solution as

$$S_t = S_0 e^{(\mu t + \sigma \int_0^t dW_t)} > 0, \quad S_0 > 0. \tag{1.4}$$

This is the stock price model used by Black and Scholes [4] in their work on the pricing of call and put options. The above equation shows that the stock price S_t is positive since $S_0 > 0$. Equation (1.3) is called geometric Brownian motion process or log-normal process. The model has an advantage over the standard Brownian motion introduced by Bachelier since it can never assume negative asset prices (see Figure 1.62).

Diffusion Processes and Stochastic Integrals

Brownian motion process W_t is continuous everywhere but integrable nowhere. Standard rules used in calculus therefore are not applicable in stochastic environment. Therefore to be able to solve the stochastic differential equation in (1.3) we introduce the Itô process.

Itô Processes

Itô process generalizes Brownian motion process (1.3) by letting parameters μ and σ depend on the underlying asset S_t and time t. Given a variable S_t, the Itô process follows the dynamics:

$$dS_t = \mu(t, S_t)dt + \sigma(t, S_t)dW_t. \tag{1.5}$$

The first term is the deterministic term which reflects the trend of the value of the underlying asset. The second term is the stochastic term.

Lemma 1.6.1. ***(Itô's Lemma)*** *Suppose the value of a variable S_t follows an Itô process*

$$dS_t = \mu(t, S_t)dt + \sigma(t, S_t)dW_t.$$

and $u(t, S_t)$ is a deterministic twice continuously differentiable function then Itô's lemma shows that:

$$du = \left(\mu S u_S + u_t + \frac{1}{2}\sigma^2 S^2 u_{SS}\right) dt + \sigma S u_S dW_t, \tag{1.6}$$

Proof. Taylor series expansion of continuous and differentiable function $u(t, S_t)$ is given by

$$\begin{aligned}\Delta u(t, S_t) &= u_t(t, S_t)\Delta t + u_S(t, S_t)\Delta S_t \\ &+ \frac{1}{2}u_{tt}(t, S_t)(\Delta t)^2 + \frac{1}{2}u_{SS}(t, S_t)(\Delta S_t)^2 \\ &+ \frac{1}{2}u_{St}(t, S_t)\Delta t(\Delta S_t) + \cdots ,\end{aligned} \tag{1.7}$$

where $u_{tt} = \frac{\partial^2 u}{\partial t^2}$ and $u_{St} = \frac{\partial^2 u}{\partial t \partial S}$. Taking the limit $\Delta t \to 0$ and applying the following informal rules

$$dt.dt = 0,\ dt.dW = 0,\ dW.dW = dt, \tag{1.8}$$

reduces the Taylor series expansion in (1.7) to

$$du = u_t dt + u_S dS_t + \frac{1}{2}u_{SS} dS_t^2. \tag{1.9}$$

From (1.3) and the informal rules in (1.8) we get

$$(dS_t)^2 = \mu^2 S_t^2 (dt)^2 + 2\mu\sigma S_t^2 dt dW_t + \sigma^2 S_t^2 (dW_t)^2 = \sigma^2 S_t^2 dt. \tag{1.10}$$

Substituting (1.3) and (1.10) into (1.9) yields

$$du = \left(u_t + v(t, S_t)u_S + \frac{1}{2}(\lambda(t, S_t))^2 u_{SS}\right) dt + \lambda(t, S_t)u_S dW_t, \tag{1.11}$$

which is the Itô's formula. □

The solution to the SDE that is equation (1.5) for the process S_t is called a *diffusion process.* Integration of (1.5) is given by

$$S_t = s + \int_0^t \mu(\tau, S_\tau)d\tau + \int_0^t \sigma(\tau, S_\tau)dW_\tau, \; s = S(0). \tag{1.12}$$

The last term in equation (1.12) is called *stochastic (Itô) integral* (see [10]).

To solve for S_t in (1.3) we apply Itô's lemma by letting $F(S_t) = \ln S_t$, $v(t, S_t) = \mu S$ and $\lambda(t, S_t) = \sigma S$.

Then from Itô's lemma we have,

$$dF = \frac{dF}{dS_t}dS_t + \frac{1}{2}\sigma^2 S_t^2 \frac{d^2F}{dS_t^2}dt$$

$$d\ln S_t = \frac{1}{S_t}dS_t - \frac{1}{2}\frac{1}{S_t^2}dS_t^2.$$

Using the informal rules in (1.8) we get

$$\begin{aligned} d\ln S_t &= \frac{1}{S_t}(\mu S_t dt + \sigma S_t dW_t) - \frac{1}{2}\frac{1}{S_t^2}S_t^2(\sigma^2 dW_t^2) \\ &= \mu dt + \sigma dW_t - \frac{1}{2}\sigma^2 dt \\ &= (\mu - \frac{1}{2}\sigma^2)dt + \sigma dW_t. \end{aligned} \tag{1.13}$$

Equation (1.13) is now integrable using standard rules in calculus unlike (1.3). Its integration therefore is

$$S_t = S_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t} > 0, \quad S_0 > 0. \tag{1.14}$$

1.7 Black-Scholes Option Pricing Theory

Standard Option Valuation Theory

The Black Scholes equation which is a Partial Differential Equation providing financial markets with a way of pricing options was obtained by considering an option maturing at time T for a non dividend paying stock. The value of an option is a function of various parameters such as strike price K and the time to expiry $T - t$ (τ), where t is the current time. It also depends on the properties of the asset itself such as its price S_t, its drift μ and its volatility σ as well as the risk free rate of interest r. The option's value can thus be written as $u(S, t)$ since μ, σ, K, T and r are parameters i.e. fixed.

The way the no-arbitrage principle is used in the derivation of the Black-Scholes equation depends on a number of assumptions about the market. First, it must be possible to buy or sell any finite quantity of the underlying security at any time (perfect liquidity). Second, security trading is continuous in time. Formally, assumptions applied in the derivation of Black Scholes equation (see [18]) are as follows:

(i) The underlying asset S_t follows geometric Brownian motion,

(ii) The rate of interest r and volatility σ are known constants over the life of the option,

(iii) There are no dividends paid during the period of the contract,

(iv) Arbitrage opportunities do not exist,

(v) There are no transaction costs on the underlying,

(vi) Trade in the underlying asset is continuous.

From the above assumptions Black and Scholes [4] form a riskless portfolio consisting of a position in the option $u(S, t)$ and a position in the underlying stock S_t. In the absence of arbitrage opportunities, the return from the portfolio must be the risk free interest rate r.

Linear Black-Scholes Model

Black and Scholes [4] assumed that the underlying asset S_t follows a geometric Brownian motion. Hence from (1.3) we have,

$$dS_t = \mu S_t dt + \sigma S_t dW_t. \tag{1.15}$$

This is the Black-Scholes model.

Linear Black-Scholes Partial Differential Equation

Let $u(S,t)$ denote the value of a European call or put option that depends only on the asset price S at time t. Let Π_t be the value of a portfolio containing one option and $-\Delta$ units of the underlying asset such that the value of the portfolio is

$$\Pi_t = u(S,t) - \Delta S. \tag{1.16}$$

From Itô's lemma we have

$$du = (u_t + \frac{1}{2}\sigma^2 S^2 u_{SS})dt + u_S dS. \tag{1.17}$$

Thus, the portfolio in (1.16) changes by

$$d\Pi_t = du(S,t) - \Delta dS$$

as Δ remains constant during the time step dt. Therefore from the equation above and (1.17) the change in the portfolio becomes

$$d\Pi_t = (u_t + \frac{1}{2}\sigma^2 S^2 u_{SS})dt + u_S dS - \Delta dS. \tag{1.18}$$

Rearranging the above equation gives:

$$d\Pi_t = (u_t + \frac{1}{2}\sigma^2 S^2 u_{SS})dt + (u_S - \Delta)dS.$$

The risk in our portfolio are the random terms

$$u_S - \Delta.$$

To eliminate the risk, we let $\Delta = u_S$ so that the randomness is reduced to zero. This is called delta hedging. Now the portfolio value changes by

$$d\Pi_t = (u_t + \frac{1}{2}\sigma^2 S^2 u_{SS})dt \tag{1.19}$$

after delta hedging. This change is completely riskless. Thus

$$d\Pi_t = r\Pi_t dt \tag{1.20}$$

where $r > 0$ is a continuously compounded interest rate. Since $\Delta = u_S$ then equation (1.16) becomes

$$\Pi_t = u(S,t) - Su_S.$$

This means that

$$d\Pi_t = r(u(S,t) - Su_S)dt \tag{1.21}$$

using equation (1.20). Comparing equations (1.19) and (1.21) and simplifying gives the linear Black Scholes PDE

$$u_t + \frac{1}{2}\sigma^2 S^2 u_{SS} + rSu_S - ru = 0,\ S > 0,\ 0 \leq t \leq T. \tag{1.22}$$

Black-Scholes Option Pricing Formulae

The Black Scholes formula for the prices at time zero of a European call option $C(S,t)$ and a European put option $P(S,t)$ on a non dividend paying stock are derived by solving the PDE in (1.22). We use boundary conditions to specify the values of the derivative at the boundaries where S and t lie. For a European call option, the boundary conditions are:

i) $C(0,t) = 0$ for $0 \leq t \leq T$,

ii) $C(S,t) \sim S - Ke^{-r(T-t)}$ as $S \to \infty$.

The pay-off function is given by

$$C(S,T) = (S_T - K)^+ = \max\{S_T - K, 0\} \text{ for } 0 \leq S$$

because it can only be exercised if $S_T > K$. The call option's value is given as

$$C(S,t) = S_t N d_1 - K e^{-r(T-t)} N d_2, \tag{1.23}$$

where

$$d_1 = \frac{\ln\left(\frac{S_t}{K}\right) + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$$

and

$$\begin{aligned} d_2 &= \frac{\ln\left(\frac{S_t}{K}\right) + (r - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} \\ &= d_1 - \sigma\sqrt{T-t}, \end{aligned} \tag{1.24}$$

$N(.)$ is the cumulative distribution function of standard normal distribution which is given by

$$\begin{aligned} N(d_1) &= \int_{-\infty}^{d_1} f(u)du \\ &= \int_{-\infty}^{d_1} \frac{1}{\sqrt{2\pi}} e^{\frac{-u^2}{2}} du, \ -\infty < d_1 < \infty. \end{aligned} \tag{1.25}$$

and $N(-x) = 1 - N(x)$. For a European put option the boundary conditions are

i) $P(t,0) = Ke^{-r(T-t)}$ for $0 \le t \le T$,

ii) $P(S,t) \to 0$ as $S \to \infty$.

The payoff function is given by

$$P(S,T) = (K - S_T)^+ = \max\{K - S_T, 0\} \text{ for } 0 \le S$$

because it can only be exercised if $K > S_T$. The put option's price is given by

$$P(S,t) = Ke^{-r(T-t)} N(-d_2) - S_t N(-d_1). \tag{1.26}$$

The Option Greeks

A financial institution that sells an option to a client in the over-the-counter markets is faced with the problem of managing its risk [18]. The Greeks are used to address this problem. Greek letters are sensitivities of the option price to a single unit change in the value of either a state variable or a parameter. Each Greek measures a different dimension to the risk in an option position and the aim of a trader is to manage the Greeks so that all risks are acceptable. These derivatives are important in the hedging of an option position, playing key roles in risk management. We shall focus on the Greek letters of European call option only.

From the Black-Scholes pricing model the price of a call option is given by

$$C(S,t) = S_t N d_1 - K e^{-r(T-t)} N d_2. \tag{1.27}$$

Delta

The *delta* of a portfolio of options (or of an option) is the sensitivity of the portfolio (or option) to the underlying asset's price. It is the rate of change of the option price with respect to the price of the underlying asset. It is the slope of the price curve of the option to the market price of the underlying asset. By calculating a *delta*, a financial institution that sells option to a client can make a delta neutral position to hedge the risk or changes of the underlying asset price. The relationship between option price and stock price is not linear so *delta* changes over different stock prices. If an investor wants to retain his portfolio in a delta neutral position, then he should adjust his hedged ratio periodically. The more frequently adjustments he does, the better delta-hedging he gets. For a European call option on a non-dividend-paying stock, *delta* is obtained as follows: First we find $N'(d_1)$

$$\frac{\partial N(d_1)}{\partial d_1} = N'(d_1) = \frac{1}{\sqrt{2\pi}} e^{\frac{-d_1^2}{2}}, \tag{1.28}$$

then

$$\begin{aligned}
\text{Delta} = \Delta = \frac{\partial C_t}{\partial S_t} &= Nd_1 + S_t\frac{\partial Nd_1}{\partial S_t} - Ke^{-r\tau}\frac{\partial Nd_2}{\partial S_t} \\
&= Nd_1 + S_t\frac{\partial Nd_1}{\partial d_1}.\frac{\partial d_1}{\partial S_t} - Ke^{-r\tau}\frac{\partial Nd_2}{\partial d_2}.\frac{\partial d_2}{\partial S_t} \\
&= Nd_1 + S_t\frac{1}{\sqrt{2\pi}}e^{\frac{-d_1^2}{2}}.\frac{1}{S_t\sigma\sqrt{\tau}} - Ke^{-r\tau}\frac{1}{\sqrt{2\pi}}e^{\frac{-d_1^2}{2}}\frac{S_t}{K}e^{r\tau}.\frac{1}{S_t\sigma\sqrt{\tau}} \\
&= Nd_1 + \frac{1}{\sigma\sqrt{\tau}}N'd_1 - \frac{1}{\sigma\sqrt{\tau}}N'd_1 \\
&= Nd_1
\end{aligned} \tag{1.29}$$

where $\tau = T - t$.

Gamma

Since the option is not linearly dependent on its underlying asset, delta-neutral hedge strategy is useful only when the movement of underlying asset price is small. Once the underlying asset price becomes wider, *gamma*-neutral hedge is necessary. An option's *gamma*, Γ, is the rate at which the delta of a portfolio (or the option) changes with respect to the underlying asset's price. For a European call option on a non-dividend paying stock *gamma* is given by,

$$\begin{aligned}
\Gamma &= N'(d_1).\frac{\partial d_1}{\partial S} \\
&= N'(d_1).\frac{\frac{1}{S}}{\sigma\sqrt{\tau}} \\
&= \frac{N'(d_1)}{S\sigma\sqrt{\tau}}.
\end{aligned} \tag{1.30}$$

Speed

The *speed* of an option is the rate of change of the gamma with respect to the stock price. For a European call option on a non-dividend paying stock *speed* is given by,

$$
\begin{aligned}
\text{Speed} &= \frac{\partial \Gamma}{\partial S} \\
&= \frac{\partial}{\partial S}\left(\frac{N'(d_1)}{S\sigma\sqrt{\tau}}\right) \\
&= -\frac{N'(d_1)}{S^2\sigma\sqrt{\tau}} + \frac{1}{S\sigma\sqrt{\tau}}.\frac{\partial N'(d_1)}{\partial S} \\
&= -\frac{N'(d_1)}{S^2\sigma^2\tau}[d_1 + \sigma\sqrt{\tau}]
\end{aligned}
\tag{1.31}
$$

Gamma is used by traders to estimate how much they will rehedge by if the stock price moves. The delta may change by less or more the amount the traders have approximated the value of the stock price to change. If it is by a large amount that the stock price moves, or the option nears the strike and expiration, the delta becomes unreliable and thus the use of speed.

Theta

Theta of portfolio of options is the rate of change of the value of the portfolio with respect to the passage of time with all else remaining the same. The value of an option is the combination of time value and stock value. When time passes the value of the option decreases. Thus the rate of change of the option price with respect to the passage of time, (i.e.*theta*) is usually negative. Theta is not directly used to hedge option position. Since there is no uncertainty to the passage of time, one does not try to hedge its effect. However, it is useful as an aid in figuring out how the value of an option depreciates as time passes and in planning for future transactions and transaction costs to keep *delta* in balance. For a European call option on a non-dividend paying stock *Theta* is given by,

$$\begin{aligned}
\text{Theta} = \Theta = \frac{\partial C}{\partial \tau} &= -S\frac{\partial N(d_1)}{\partial \tau} + (-r)Ke^{-r\tau}N(d_2) + Ke^{-r\tau}\frac{\partial N(d_2)}{\partial \tau} \\
&= -S\frac{\partial N(d_1)}{\partial d_1}.\frac{\partial d_1}{\partial \tau} - rKe^{-r\tau}N(d_2) + Ke^{-r\tau}\frac{\partial N(d_2)}{\partial d_2}.\frac{\partial d_2}{\partial \tau} \\
&= -S\frac{1}{\sqrt{2\pi}}e^{\frac{-d_1^2}{2}}\left(\frac{\frac{\sigma^2}{2}}{\sigma\sqrt{\tau}}\right) - rKe^{-r\tau}N(d_2) \\
&= -SN'(d_1)\frac{\sigma S}{2\sqrt{\tau}} - rKe^{-r\tau}N(d_2).
\end{aligned} \tag{1.32}$$

Rho

The rho of a portfolio of options is the rate of change of the value of the portfolio with respect to the interest rate. The *rho* for an ordinary stock call option should be positive because higher interest rate reduces the present value of the strike price which in turn increases the value of the call option. For a European call option on a non-dividend paying stock *rho* is given by,

$$\begin{aligned}
\text{Rho} = \rho = \frac{\partial \Pi}{\partial r} &= \frac{\partial C}{\partial r} \\
&= S\frac{\partial N(d_1)}{\partial r} - (-\tau)Ke^{-r\tau}N(d_2) - Ke^{-r\tau}\frac{\partial N(d_2)}{\partial r} \\
&= SN'(d_1)\frac{\sqrt{\tau}}{\sigma} + K\tau e^{-r\tau}N(d_2) - SN'(d_1)\frac{\sqrt{\tau}}{\sigma} \\
&= K\tau e^{-r\tau}N(d_2).
\end{aligned} \tag{1.33}$$

Vega

It is also referred to as *kappa* or *zeta*. This is completely different from other Greeks since it is a derivative with respect to a parameter and not a variable. During the derivation of the Black-Scholes formula, the volatility σ of the asset underlying a derivative is assumed to be constant. In reality, volatilities change over time. This means that the value of a derivative is liable to change because of movements in volatility as well as because of change in the asset price and passage of time. For a European call option on a non-dividend paying stock *vega* is given by,

$$\begin{aligned}\text{Vega} = \nu &= \frac{\partial C}{\partial \sigma} \\ &= S\frac{\partial N(d_1)}{\partial \sigma} - Ke^{-r\tau}\frac{\partial N(d_2)}{\partial \sigma} \\ &= S\frac{\partial N(d_1)}{\partial d_1}.\frac{\partial d_1}{\partial \sigma} - Ke^{-r\tau}\frac{\partial N(d_2)}{\partial d_2}.\frac{\partial d_2}{\partial \sigma} \\ &= SN'(d_1)\sqrt{\tau}.\end{aligned} \tag{1.34}$$

Modified Option Valuation Theory

Transaction-Cost Model

We consider a market with one share denoted by S_t, and a risk free money market account with spot rate of interest $r \geq 0$ whose value at time t is $B_t \equiv 1$. Stock is illiquid (its price is affected by trading) while money market account is assumed to be liquid. The model we are going to focus on is the model due to Cetin *et al.* [9] where a fundamental stock price S_t^0 follows the dynamics

$$dS_t^0 = \mu S_t^0 + \sigma S_t^0 dW_t,\ 0 \leq t \leq T.$$

An investor who wants to trade α shares at time t has to pay the transaction price $\overline{S}_t$ which is given by

$$\overline{S}_t(\alpha) = e^{\rho\alpha} S_t^0$$

where ρ is the liquidity parameter with $\rho \geq 0$. This models a bid-ask-spread (the amount by which the offer price exceeds the bid price) whose size depends on α (number of shares traded).

Consider a Markovian trading strategy (i.e. a strategy of the form $\Phi_t = \phi(t, S_t^0)$) for a smooth function $\phi = u_S$ where ϕ is the hedge ratio. Then we have $\phi_S = u_{SS}$. If the stock and bond positions are Φ_t and η_t respectively then the value of this strategy at time t is

$$V_t = \Phi_t S_t^0 + \eta_t.$$

Φ_t is a semi-martingale with quadratic variation of the form

$$[\Phi]_t = \int_0^t (\phi_S(\tau, S_\tau^0)\sigma S_\tau^0)^2 d\tau$$

whose change is given by $d[\Phi]_t = (u_{SS}(t, S_t^0)\sigma S_t^0)^2$ since $\phi_S = u_{SS}$.

Applying Itô formula to $u(t, S_t^0)$ gives

$$du(t, S_t^0) = u_S(t, S_t^0)dS_t^0 + \left(u_t(t, S_t^0) + \frac{1}{2}\sigma^2(S_t^0)^2 u_{SS}(t, S_t^0)\right) dt \tag{1.35}$$

For a continuous semi-martingale Φ with quadratic variation $[\Phi]_t$ the wealth dynamics of a self-financing strategy becomes

$$dV_t = \Phi_t dS_t^0 - \rho S_t^0 d[\Phi]_t. \tag{1.36}$$

Substituting $d[\Phi]_t$ into (1.36) yields the following dynamics:

$$dV_t = \phi(t, S_t^0)dS_t^0 - \rho S_t^0(\phi_S(t, S_t^0)\sigma S_t^0)^2 dt. \tag{1.37}$$

Since $V_t = u(t, S_t^0)$, equating the deterministic components of (1.35) and (1.37) and taking $\phi_S = u_{SS}$ gives the nonlinear PDE

$$u_t + \frac{1}{2}\sigma^2 S^2 u_{SS}(1 + 2\rho S u_{SS}) = 0, \; u(S_T^0) = h(S_T^0) \tag{1.38}$$

where $h(S_T^0)$ is the payoff of the value claim at maturity time T.

Bakstein and Howison (2003) Model

This is a model of illiquid markets which results in the PDE

$$u_t + \frac{1}{2}\sigma^2 S^2 u_{SS}(1 + 2\rho S u_{SS}) + \frac{1}{2}\rho^2(1-\alpha)^2\sigma^2 S^4 u_{SS}^3 + rSu_S - ru = 0 \tag{1.39}$$

where ρ is the liquidity parameter of the market and α is a measure of the price slippage impact of a trade felt by all market participants (see [2]). When $\alpha = 1$, this corresponds to no slippage and equation (1.39) reduces to the PDE

$$u_t + \frac{1}{2}\sigma^2 S^2 u_{SS}(1 + 2\rho S u_{SS}) + rSu_S - ru = 0 \tag{1.40}$$

This is the Cetin *et al.* model given in (1.38) with $r > 0$.

Solution of the Nonlinear Black-Scholes Equation

Theorem 1.7.1. *If $V(x,t)$ is any positive solution to the porous medium type equation*

$$V_t + \frac{\sigma^2}{2}(VV_x + \frac{1}{2}V^2)_x = 0,$$

then

$$u(S,t) = S - \frac{\sqrt{S_0}}{\rho}\left(\sqrt{S}e^{\frac{\sigma^2 t}{8}} + \frac{\sqrt{S_0}}{4}e^{\frac{\sigma^2 t}{4}}\right) \tag{1.41}$$

solves the nonlinear Black-Scholes equation $u_t + \frac{1}{2}\sigma^2 S^2 u_{SS}(1+2\rho S u_{SS}) = 0$ *for* $S_0, S, \sigma, \rho > 0$ *and* $t \geq 0$.

Equation (1.41) is the formula for the European call option with $r = 0$ with the proof of this thorem found in [13].

Theorem 1.7.2. *If $V(x,t)$ is any positive solution to the porous medium tyupe equation*

$$V_t + \frac{\sigma^2}{2}(VV_x + \frac{1}{2}V^2 + \frac{2r}{\sigma^2}V)_x = 0,$$

then

$$u(S,t) = S - \frac{\sqrt{S_0}}{\rho}\left(\sqrt{S}e^{(\frac{r+\frac{\sigma^2}{4}}{2})t} + \frac{\sqrt{S_0}}{4}e^{(r+\frac{\sigma^2}{4})t}\right) \tag{1.42}$$

solves the nonlinear Black-Scholes equation $u_t + \frac{1}{2}\sigma^2 S^2 u_{SS}(1+2\rho S u_{SS}) + rSu_S - ru = 0$ *for* $S_0, S, \sigma, \rho > 0$ *and* $r, t \geq 0$.

Equation (1.42) is the formula for the European call option with $r \geq 0$ with the proof of this theorem found in [12].

CHAPTER 2

Literature Review

In this chapter we give a brief history on stock price modelling from the time Brownian motion was discovered until the time the linear Black-Scholes equation was derived and solved. This will involve the study that has so far been carried out on Greek parameters. We also mention some of the nonlinear Black-Scholes models which have been studied.

2.1 Brownian Motion

The discovery of Brownian motion is usually credited to the Scottish botanist Robert Brown. Brown [7] observed the random motion of pollen grains suspended in water. Brown described the observed movement but did not have a scientific explanation for the observed phenomenon. Bachelier [1], on the other hand, introduced Brownian motion as a model for the dynamic behavior of the stock market in his Ph. D thesis titled "The Theory of Speculation". Bachelier used the Brownian process to model a European type option of traded asset prices. His work was remarkable because by addressing the problem of option pricing, Bachelier derived most of the theory of diffusion processes. The major weakness with Bachelier's work was that the model could take a negative asset price and using it directly to model stock prices was questionable.

Five years later, Albert Einstein [11] worked independently and discovered the same stochastic process and applied it in thermodynamics. He introduced the diffusion coefficient which is called diffusivity in physics commonly known as volatility in price dynamics. Norbert Wiener proved the existence of Brownian motion and gave the first mathematical construction of Brownian motion. That is why Brownian motion is sometimes called the Wiener process. Samuelson [27] introduced a non negative variation of Brownian motion called Geometric Brownian motion. He modified Bachelier's model assuming that the return rates instead of stock prices as was done by Bachelier follow a geometric Brownian motion. As a result of the geometric Brownian motion the stock prices follow a lognormal distribution eliminating the problem of negative stock price as observed in Bachelier's model.

2.2 Diffusion Processes and Stochastic Integrals

Brownian motion is continuous everywhere but nowhere integrable almost surely (see Theorem VII of Paley *et al* [25]). Ordinary rules of calculus fail in a stochastic environment due to the stochastic component. In 1944, Kiyoshi Itô [19] went on to develop stochastic calculus, the machinery needed in order to use Brownian motion to model stock prices successfully and which later became an essential tool of financial mathematics.

2.3 Standard Option Pricing Theory

In 1973, research picked up when Black and Scholes [4] came up with the famous Black-Scholes equation. The equation was obtained from the Black-Scholes model (i.e. geometric Brownian motion). The equation was solved to obtain the Black-Scholes formulae for pricing European call and put options for a non-dividend paying stock. Since the derivation of this model, much of the work undertaken in mathematical finance has been aimed at relaxing a number of modeling assumptions. One of the assumptions was that the market in the underlying asset is perfectly liquid such that trading has no impact on the price of the underlying.

2.4 Modified Option Pricing Theory

Relaxing any of the assumptions made by Black and Scholes [4] used in the classical theory leads to modified nonlinear equations. One of the assumption was that volatility was constant but Otula in [24] incorporated the stochastic nature of volatility and derived a logistic Black-Scholes-Merton Partial Differential Equation and came up with the equation

$$u_t + \frac{1}{2}\sigma^2 S^2 (S^* - S)^2 u_{SS} + \sigma^2 S(S^* - S)\upsilon_\sigma u_{S\sigma} + \frac{1}{2}\upsilon_\sigma^2 \sigma^2 u_{\sigma\sigma} + rS(S^* - S)u_S + (\mu_\sigma - \lambda\upsilon_\sigma) - ru = 0 \tag{2.1}$$

where $u_t = \frac{\partial u}{\partial t}$, $u_S = \frac{\partial u}{\partial S}$, $u_{SS} = \frac{\partial^2 u}{\partial S^2}$, $u_{S\sigma} = \frac{\partial^2 u}{\partial S \partial \sigma}$, $u_{\sigma\sigma} = \frac{\partial^2 u}{\partial \sigma^2}$, $u(S,t)$ is the price of a call option, σ is the volatility, S is the stock price, r is the risk free interest rate, S^* is the Walrasian equilibrium market price, λ is the market price of volatility risk, υ_σ is the variance of asset volatility, μ_σ is the mean asset volatility and $\mu_\sigma - \lambda\upsilon_\sigma$ the risk neutral drift rate of volatility. υ_σ has not been explicitly expressed, therefore its difficult to get an analytic solution to the equation above.

Rangita in [26] developed and solved both deterministic and stochastic differential equations for stock price, unlike PDEs which can be solved and the solutions are partially differentiated to give the Greek parameters. Brian in [6] used numerical approach to estimate market volatility using logistic Brownian motion which gives approximate values unlike analytic methods which gives exact values.

Relaxing the liquidity assumption leads to nonlinear behavior. Work that has led to such class of nonlinear PDEs in finance to date include the model of Leland [20] which was the groundwork for modeling the effects of transaction costs. Leland adopted the hedging strategy of rehedging at every time step δt where δt is a finite, fixed and small time step. He assumed that the transaction cost $\kappa|\Delta|S/2$, where κ denotes the round trip transaction cost per unit dollar of the transaction and Δ the number of assets bought ($\Delta > 0$) or sold ($\Delta < 0$) at price S, is proportional to the monetary value of the assets bought or sold. Leland showed that

$$\frac{\kappa}{2}|\delta\Delta|S = \frac{\sigma^2}{2}\mathrm{Le}S^2|u_{SS}|\delta t,$$

where Le denotes the Leland number which is given by

$$\mathrm{Le} = \sqrt{\frac{2}{\pi}}\left(\frac{\kappa}{\sigma\sqrt{\delta t}}\right)$$

with δt being the transaction frequency. Hoggard-Whalley-Willmott (1992) (see [28]) model is a simple transaction cost model for non-vanilla options and option portfolios based on Leland's hedging strategy and model. Leland's equation is given by

$$u_t + \frac{1}{2}\sigma^2 S^2 u_{SS} - \kappa\sigma S^2\sqrt{\frac{2}{\pi\delta t}}|u_{SS}| + rSu_S - ru = 0, \tag{2.2}$$

where $\kappa\sigma S^2\sqrt{\frac{2}{\pi\delta t}}|u_{SS}|$ is the expected transaction cost over a time step δt. This was one of the first nonlinear PDEs in the theory of derivatives. The main model in this class was put forward by Cetin *et al.* [9] where transaction cost was shown to be proportional to the quadratic variation of the stock trading strategy.

2.5 Risk Parameters

The risk parameters (commonly known as the Greeks) are used by financial institutions which sell options to hedge risks. The risk parameter *delta* was used by Black and Scholes [4] for hedging European call options. Since then, other risk parameters such as *gamma, speed, vega, rho* and *theta* have been computed from the Black-Scholes formulae. These Greek parameters were computed under the assumption that the market was perfectly liquid. Relaxing this assumption brings in the issue of liquidity risk. To explain this risk, we need to compute the Greek parameters from the formula of a European call option of a nonlinear Black-Scholes equation whose nonlinearity is as a result of transaction costs. The Greek parameters arising from the nonlinear equation when $r > 0$ have been computed by Esekon in [12, 15] while those with $r = 0$ for a particular solution of a nonlinear equation have not been computed. Therefore, we computed the Greek parameters with $r = 0$ and compared them with those when $r > 0$.

CHAPTER 3

Results and Discussion

In this chapter we compute the risk parameters resulting from the solutions of the nonlinear Black-Scholes equations for $r = 0$ and $r > 0$. We then compare these risk parameters by plotting the risk parameters against the prices of the stock. The stock price data from the Nairobi Securities Exchange is used to plot the curves of the risk parameters derived from the solution of the nonlinear Black-Scholes equation to test whether the risk parameters are applicable in a real life situation. A subset of data for KPLC and KenGen for the periods of 3rd January 2005 - 3rd January 2006 and 2nd June 2006 - 5th June 2007 respectively were used.

3.1 Greek Parameters of a Nonlinear Black-Scholes Equation

To obtain the Greek parameter *delta* we differentiate the solution of the European call option with respect to the spatial variable S. For $r = 0$ we differentiate equation (1.41) with respect to S to obtain

$$u_S = 1 - \frac{1}{2\rho}\sqrt{\frac{S_0}{S}}e^{\frac{\sigma^2 t}{8}} \tag{3.1}$$

for $\rho, S, \sigma > 0$ and $S_0 \geq 0, t \geq 0$. For $r > 0$ we use equation (1.42) to obtain

$$u_S = 1 - \frac{1}{2\rho}\sqrt{\frac{S_0}{S}}e^{\left(\frac{r+\frac{\sigma^2}{4}}{2}\right)t} \tag{3.2}$$

for $\rho, S, \sigma > 0$ and $S_0 \geq 0, r \geq 0, t \geq 0$.

To obtain the Greek parameter *gamma* for $r = 0$ we differentiate equation (3.1) with respect to S to get

$$u_{SS} = \frac{1}{4\rho S^{\frac{3}{2}}}\sqrt{S_0}e^{\frac{\sigma^2 t}{8}} \tag{3.3}$$

for $\rho, S_0, S, \sigma > 0$ and $t \geq 0$. For $r > 0$ we use equation (3.2) to obtain

$$u_{SS} = \frac{1}{4\rho S^{\frac{3}{2}}}\sqrt{S_0}e^{\left(\frac{r+\frac{\sigma^2}{4}}{2}\right)t} \tag{3.4}$$

for $\rho, S_0, S, \sigma > 0$ and $r \geq 0, t \geq 0$.

Differentiating equation (3.3) and (3.4) with respect to the spatial variable S gives the Greek parameter *speed*. For $r = 0$ it is given by

$$u_{SSS} = -\frac{3}{8\rho S^{\frac{5}{2}}}\sqrt{S_0}e^{\frac{\sigma^2 t}{8}} \tag{3.5}$$

for $\rho, S_0, S, \sigma > 0$ and $t \geq 0$. Similarly, for $r > 0$ *speed* is given by

$$u_{SSS} = -\frac{3}{8\rho S^{\frac{5}{2}}}\sqrt{S_0}e^{\left(\frac{r+\frac{\sigma^2}{4}}{2}\right)t} \tag{3.6}$$

for $\rho, S_0, S, \sigma > 0$ and $r \geq 0, t \geq 0$.

When we differentiate the solution (1.42) with respect to r we get the Greek parameter *rho* as

$$u_r = -t\frac{\sqrt{S_0}}{2\rho}\left\{\sqrt{S}e^{\left(\frac{r+\frac{\sigma^2}{4}}{2}\right)t} + \frac{\sqrt{S_0}}{2}e^{\left(r+\frac{\sigma^2}{4}\right)t}\right\} \tag{3.7}$$

for $\rho, S_0, S, \sigma > 0$ and $r \geq 0, t \geq 0$.

Differentiating the solution of the European call option $u(S,t)$ with respect to the parameter σ gives the Greek parameter *vega*. To obtain *vega* for $r = 0$ we differentiate equation (1.41) with respect to σ to obtain

$$u_\sigma = -\frac{\sigma t\sqrt{S_0}}{4\rho}\left\{\sqrt{S}e^{\frac{\sigma^2 t}{8}} + \frac{\sqrt{S_0}}{2}e^{\frac{\sigma^2 t}{4}}\right\} \tag{3.8}$$

for $\rho, S_0, S, \sigma > 0$ and $t \geq 0$. To obtain *vega* for $r > 0$ we differentiate equation (1.42) with respect to σ to get

$$u_\sigma = -\frac{\sigma t\sqrt{S_0}}{4\rho}\left\{\sqrt{S}e^{\left(\frac{r+\frac{\sigma^2}{4}}{2}\right)t} + \frac{\sqrt{S_0}}{2}e^{\left(r+\frac{\sigma^2}{4}\right)t}\right\} \tag{3.9}$$

for $\rho, S_0, S, \sigma > 0$ and $r \geq 0, t \geq 0$.

Lastly, to obtain the Greek parameter *theta* we differentiate the solution $u(S,t)$ with respect to time t. For $r = 0$ we differentiate equation (1.41) with respect to t to get

$$u_t = -\frac{\sigma^2\sqrt{S_0}}{8\rho}\left\{\sqrt{S}e^{\frac{\sigma^2 t}{8}} + \frac{\sqrt{S_0}}{2}e^{\frac{\sigma^2 t}{4}}\right\} \tag{3.10}$$

for $\rho, S_0, S, \sigma > 0$ and $t \geq 0$. Differentiating equation (1.42) with respect to t gives *theta* for $r > 0$. This is given by

$$u_t = -\frac{(r + \frac{\sigma^2}{4})\sqrt{S_0}}{2\rho}\left\{\sqrt{S}e^{\left(\frac{r+\frac{\sigma^2}{4}}{2}\right)t} + \frac{\sqrt{S_0}}{2}e^{(r+\frac{\sigma^2}{4})t}\right\} \tag{3.11}$$

for $\rho, S_0, S, \sigma > 0$ and $r \geq 0, t \geq 0$.

3.2 Application of the Greeks

By letting $\omega = \frac{1}{2\rho}\sqrt{\frac{S_0}{S}}e^{\frac{\sigma^2 t}{8}}$ equations (3.1) and (3.2) can be written in terms of ω as $u_S = 1 - \omega$ and $u_S = 1 - \omega e^{\frac{rt}{2}}$ respectively. Theoretically, comparing equations (3.1) and (3.2), the Greek parameter delta when $r = 0$ is greater than when $r > 0$ because $e^{\frac{rt}{2}} > 1$ and since $\omega > 0$.

Our results from the theoretical values and Figure 3.21 shows that the delta values when parameter r is not included (Cetin *et al.* model) are higher compared to delta values when $r > 0$ (Bakstein and Howison (2003) model). Also the delta values increase with an increase in the stock price (see Figure 3.21). Delta values were positive which means that the options value will increase when the underlying stock increases and will decrease when the stock price decreases (also known as a positive relationship).

Equations (3.3) and (3.4) can be written in terms of ω as $u_{SS} = \frac{\omega}{2S}$ and $u_{SS} = \frac{\omega}{2S}e^{\frac{rt}{2}}$ respectively. Gamma is an estimate of how much the delta of an option changes when the price of the stock moves. As a tool, gamma can tell you how stable the risk parameter

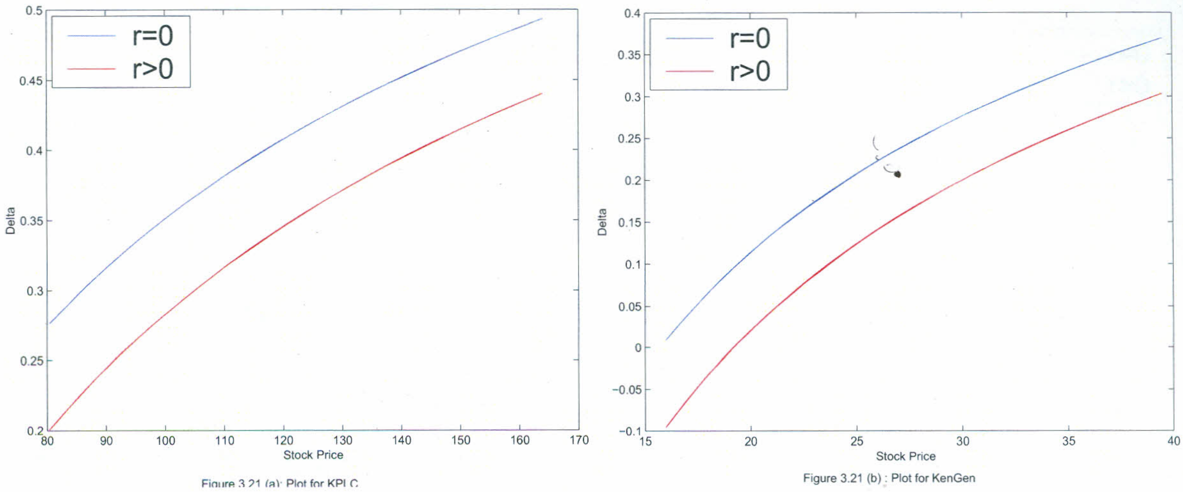

Figure 3.21: Variation of *delta* u_S with stock price S_t for a European call option when $r = 0$ and $r > 0$ for KPLC and Kengen with $S_0 = 94.5$ (KPLC), $S_0 = 35.25$ (KenGen), $\sigma = 0.1, t = 1, r = 0.2$ and $\rho = 0.75$.

delta is. The values of the Greek parameter *gamma* when $r > 0$ is greater compared to its values when $r = 0$ because $e^{\frac{rt}{2}} > 1$. Figure 3.22 shows that the Greek parameter *gamma* reduces with an increase in the stock price. Gamma values were positive which means delta will increase when the underlying stock price increases and will decrease when the stock price decreases. This is called a positive relationship. Positive Gamma makes delta more and more positive as the stock rises.

Equations (3.5) and (3.6) can be written in terms of ω as $u_{SSS} = -\frac{3\omega}{4S^2}$ and $u_{SSS} = -\frac{3\omega}{4S^2}e^{\frac{rt}{2}}$ respectively. This means that the Greek parameter speed when $r = 0$ is greater than when $r > 0$. From Figure 3.23, the Greek parameter speed increases with an increase in stock price. Also, speed values are higher in the Cetin *et al.* model compared to the Bakstein and Howison (2003) model as seen in Figure 3.23.

Writing equation (3.7) in terms of ω gives $u_r = -t\left(S\omega e^{\frac{rt}{2}} + \frac{\sqrt{S_0}}{4\rho}e^{(r+\frac{\sigma^2}{4})t}\right)$. In the Howison and Bakstein (2003) model, the risk parameter *rho* decreases with an increase in the

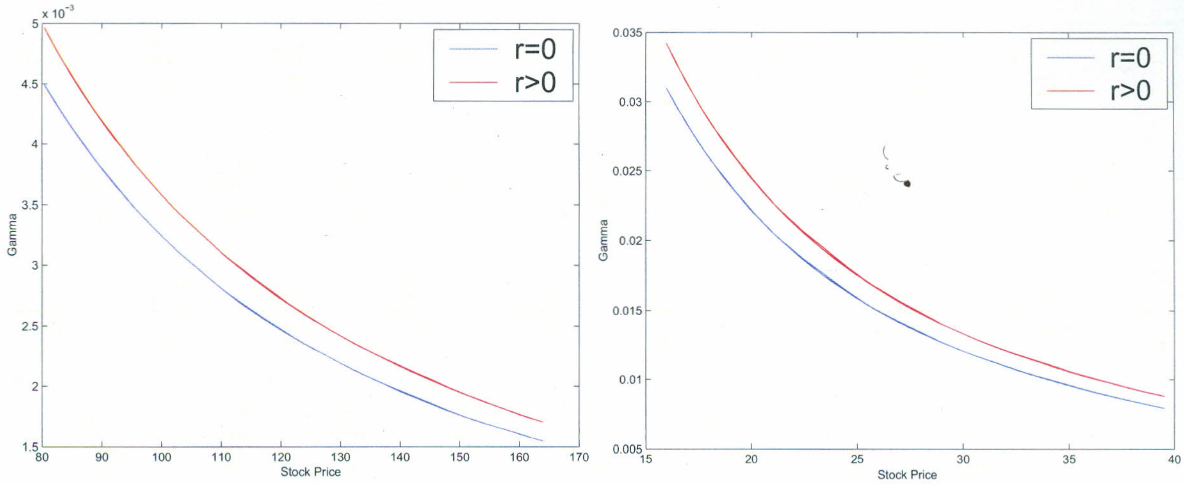

Figure 3.22 (a): Plot for KPLC

Figure 3.22 (b): Plot for KenGen

Figure 3.22: Variation of *gamma* u_{SS} with stock price S_t for a European call option when $r = 0$ and $r > 0$ for KPLC and Kengen with $S_0 = 94.5$ (KPLC), $S_0 = 35.25$ (KenGen), $\sigma = 0.1, t = 1, r = 0.2$ and $\rho = 0.75$.

stock price (see Figure 3.24). Interest rates are normally stable, therefore the chance that option's price will change drastically due to a rise or a drop in interest will be quite low. Rho values were negative which means the option's price decreases when interest rate increases and increases when the interest rate decreases.

Equations (3.8) and (3.9) in terms of ω are given by $u_\sigma = -\dfrac{\sigma t}{4}\left(S\omega + \dfrac{S_0}{2\rho}e^{\frac{\sigma^2 t}{4}}\right)$ and $u_\sigma = -\dfrac{\sigma t}{4}\left(S\omega e^{\frac{rt}{2}} + \dfrac{S_0}{2\rho}e^{(r+\frac{\sigma^2}{4})t}\right)$ respectively. This means that the Greek parameter *vega* is greater when $r = 0$ than when $r > 0$ since $e^{\frac{rt}{2}} > 1$ (see also Figure 3.25). From figure 3.25, the vega values decrease with an increase in the stock price. Higher volatility implies greater expected fluctuations in the stock price which means a greater possibility for an option to move into your favor by the expiration date. Decreased volatility means further stock price fluctuations is expected to be lower. Higher volatility leads to higher option prices. Therefore, vega can move even without any changes in the underlying stock prices.

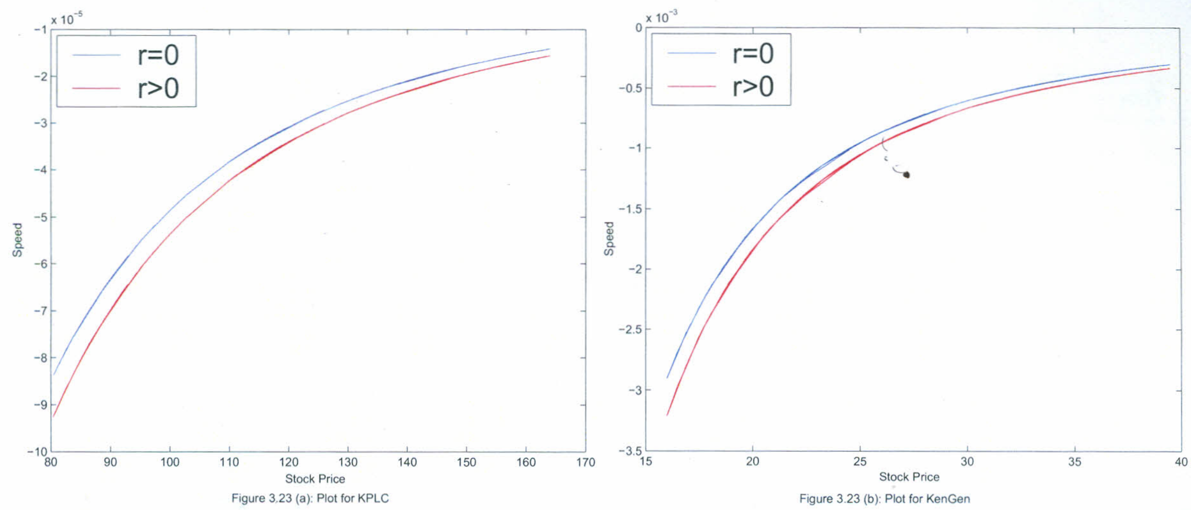

Figure 3.23: Variation of *speed* u_{SSS} with stock price S_t for a European call option when $r = 0$ and $r > 0$ for KPLC and Kengen with $S_0 = 94.5$ (KPLC), $S_0 = 35.25$ (KenGen), $\sigma = 0.1, t = 1, r = 0.2$ and $\rho = 0.75$.

Equations (3.10) and (3.11) can be written in terms of ω as $u_t = -\frac{\sigma^2}{4}\left(S\omega + \frac{S_0}{4\rho}e^{\frac{\sigma^2 t}{4}}\right)$ and $u_t = -\frac{(4r+\sigma^2)}{4}\left(S\omega e^{\frac{rt}{2}} + \frac{S_0}{4\rho}e^{(r+\frac{\sigma^2}{4})t}\right)$ respectively. This means that when $r = 0$ the Greek parameter theta is greater compared to its value when $r > 0$ (see Figure 3.26). From Figure 3.26 Theta values decrease with an increase in the stock price.Theta has much more impact on an option that is nearing expiration than an option that is far away from expiration. Theta values were negative which means that an option's value will fall as time passes.

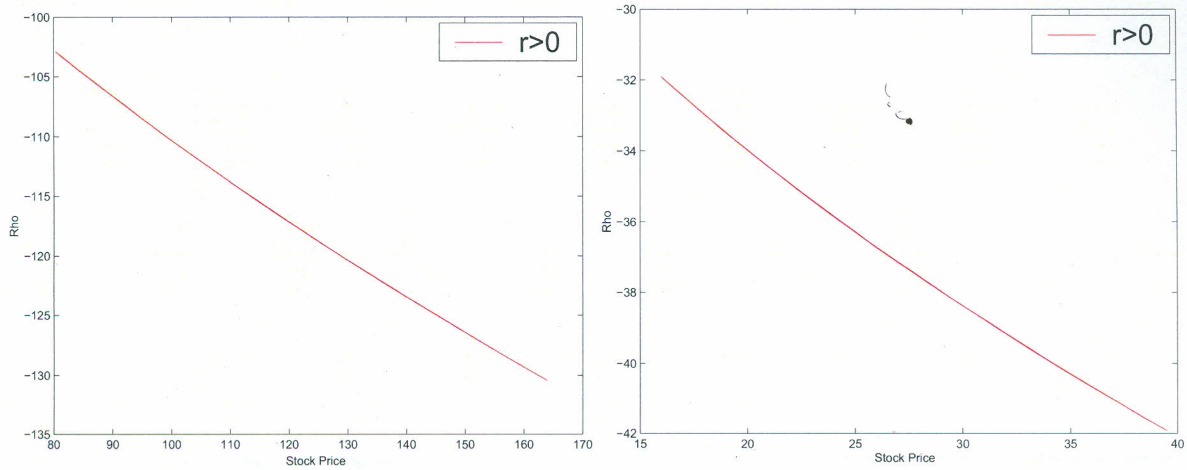

Figure 3.24 (a): Plot for KPLC

Figure 3.24 (b): Plot for KenGen

Figure 3.24: Variation of *rho* u_r with stock price S_t for a European call option when $r > 0$ for KPLC and Kengen with $S_0 = 94.5$ (KPLC), $S_0 = 35.25$ (KenGen), $\sigma = 0.1, t = 1, r = 0.2$ and $\rho = 0.75$.

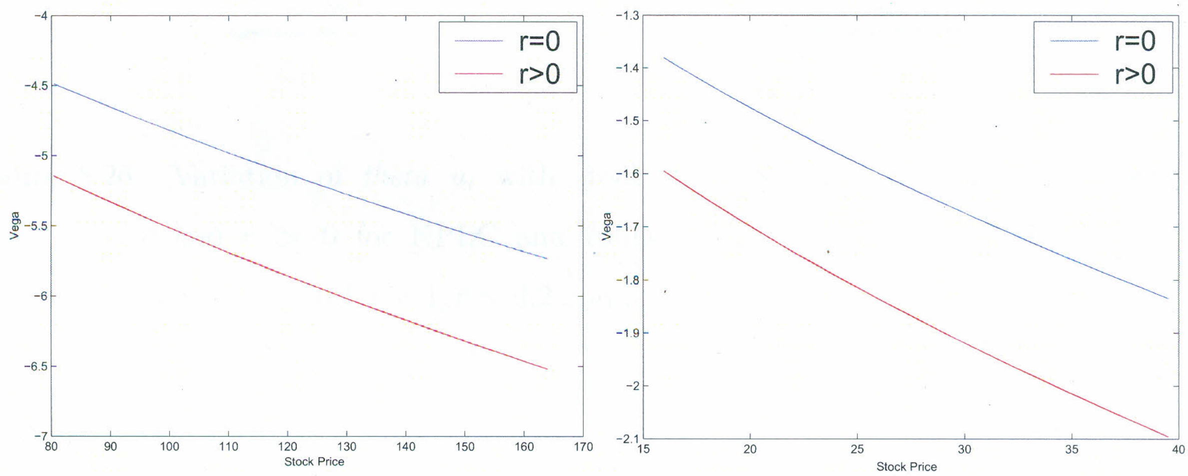

Figure 3.25 (a): Plot for KPLC

Figure 3.25 (b): Plot for KenGen

Figure 3.25: Variation of *vega* u_σ with stock price S_t for a European call option when $r = 0$ and $r > 0$ for KPLC and Kengen with $S_0 = 94.5$ (KPLC), $S_0 = 35.25$ (KenGen), $\sigma = 0.1, t = 1, r = 0.2$ and $\rho = 0.75$.

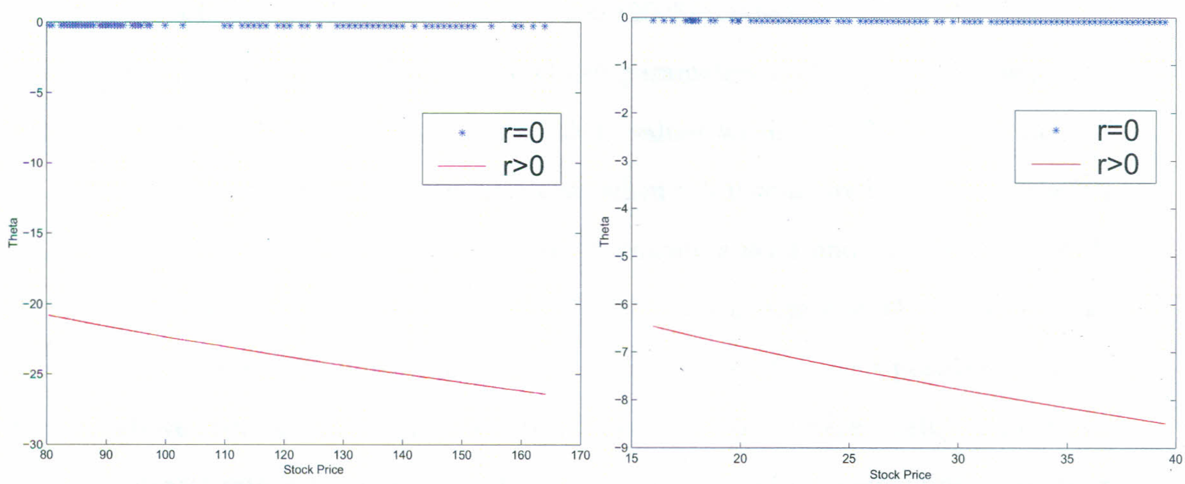

Figure 3.26 (a): Plot for KPLC

Figure 3.26 (b): Plot for KenGen

Figure 3.26: Variation of *theta* u_t with stock price S_t for a European call option when $r = 0$ and $r > 0$ for KPLC and Kengen with $S_0 = 94.5$ (KPLC) $S_0 = 35.25$ (KenGen), $\sigma = 0.1, t = 1, r = 0.2$ and $\rho \doteq 0.75$.

Summary, Conclusion and Recommendation

This thesis deals with the Greek parameters of a European call option which have been derived from a nonlinear Black-Scholes formula. These risk parameters are for a call option in illiquid markets whose illiquidity is arising from transaction costs. We have computed the Greek parameters derived from a nonlinear Black-Scholes formula in (1.41)(Cetin *et al.* model). We have compared these Greeks with those derived from a nonlinear Black-Scholes formula in (1.42) (Bakstein and Howison (2003) model).

We have found out that the values of the Greek parameters *delta, speed, rho, vega* and *theta* are higher when $r = 0$ compared to their values when $r > 0$. In contrast, the values of the Greek parameter *gamma* is higher when $r > 0$ compared to its values when $r = 0$. We have also found out that the Greek parameters *delta* and *speed* increase with an increase in the stock price while the rest (*gamma, rho, vega* and *theta*) decrease as the stock price increases. It is only the *delta* and *gamma* that were positive in value. From the above results, liquidity risk is higher when $r = 0$ (i.e. interest rate) and is lower when the interest rate is positive. Liquidity risk is usually higher in emerging markets or low volume markets. There are many factors that affect an option's price and the Greeks help us understand this process better. They show how to protect one's position against adverse movements in critical market variables such as the stock price, volatility, time and interest rate. It's possible for some Greeks to be working for ones position while others could be simultaneously working against it. Understanding how changing conditions can affect options trades, may help a trader to better position himself accordingly. All the Greek parameters are of the form $a + \frac{1}{\rho} f(S,t)$ where $a \in \mathbb{R}$.

The current work could be extended by comparing the Greek parameters of a European call option with those of European put option in the case of modified option valuation.

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