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Norm of a Derivation and Hyponormal operators

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Abstract

We characterize when the norm of inner derivation on a norm ideal equals that on the quotient algebra. We further investigate norms of inner derivations implemented by normal and hyponormal operators on norm ideals.

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1 Introduction

Let H be a complex Hilbert space and $B(H)$ the algebra of all bounded linear operators on H. A (symmetric) norm ideal $(\mathfrak{J}, \|.\|_{\mathfrak{J}})$ in $B(H)$ consists of a proper two-sided ideal $\mathfrak{J}$ together with a norm $\|.\|_{\mathfrak{J}}$ satisfying the following conditions;

- $(\mathfrak{J}, \|.\|_{\mathfrak{J}})$ is a Banach space.
- $\|AXB\|_{\mathfrak{J}} \leq \|A\|\|X\|_{\mathfrak{J}}\|B\|$ for all $X \in \mathfrak{J}$ and all operators A and B in $B(H)$
- $\|X\|_{\mathfrak{J}} = \|X\|$ for X a rank one operator.

For $A \in B(H)$, the inner derivation induced by A is the operator Δ_A defined on $B(H)$ by $\Delta_A(X) = AX - XA$, for all $X \in B(H)$. The norm of an inner derivation Δ_A on $B(H)$ is

$$\|\Delta_A|B(H)\| = 2d(A) \tag{1}$$

where $d(A) = \inf\{\|A - \lambda\| : \lambda \in \mathbb{C}\}$. See [11].
In fact, for a normed algebra, each inner derivation is bounded and $\|\Delta_A\| \leq 2\|A\|$, ([3]). Furthermore, if $\mathfrak{J}$ is a closed norm ideal in a normed algebra $B(H)$, then $B(H)/\mathfrak{J}$ is a normed algebra when multiplication is defined as $(X + \mathfrak{J})(Y + \mathfrak{J}) = (XY + \mathfrak{J})$ and $B(H)/\mathfrak{J}$ is endowed with the quotient norm; $\|X + \mathfrak{J}\| = \inf\{\|X + K\| : K \in \mathfrak{J}\}$. For an intensive study of quotient spaces see [7].
We say that a bounded linear operator A on a Hilbert space H is hyponormal if $A^*A - AA^* \geq 0$, while A is normal if $A^*A = AA^*$. Let $(\mathfrak{J}, \|.\|_{\mathfrak{J}})$ be a norm ideal and let $A \in B(H)$. If $X \in \mathfrak{J}$, then $\Delta_A(X) \in \mathfrak{J}$ and $\|AX - XA\|_{\mathfrak{J}} = \|(A - \lambda)X - X(A - \lambda)\|_{\mathfrak{J}} \leq 2\|A - \lambda\|\|X\|_{\mathfrak{J}}$ for all $\lambda \in \mathbb{C}$. Hence $\|\Delta_A(X)\|_{\mathfrak{J}} \leq 2d(A)\|X\|_{\mathfrak{J}}$, implying that

$$\|\Delta_A|\mathfrak{J}\| \leq 2d(A). \tag{2}$$

In order to examine the extent to which the identity (1) applies, L. Fialkow [4] introduced the notion of S - universal operators. An operator $A \in B(H)$ is S - universal if $\|\Delta_A|\mathfrak{J}\| = 2d(A)$ for each norm ideal $\mathfrak{J}$ in $B(H)$.
In section 2, we investigate whether the inequality (2) is true for the quotient algebra $B(H)/\mathfrak{J}$. Moreover, we apply the concept of S - universality to explore the relationship between the norm of inner derivation on a norm ideal $\mathfrak{J}$ and that on the quotient algebra $B(H)/\mathfrak{J}$.
Section 3 considers operators belonging to the Hilbert - Schmidt class $C_2(H)$. This class represents a Hilbert space when equipped with the inner product $\langle X, Y\rangle = tr(XY^*)$, $(X, Y \in C_2(H))$, where tr stands for the usual trace functional and Y^* denotes the adjoint of Y. Finally we establish the condition for which the equality $\|\Delta_N|C_2\| = \|\Delta_A|C_2\|$ holds true, with A, N being arbitrary hyponormal and normal operators respectively.

2 Norm Ideals and S - universality

For a complete account of the theory of norm ideals, we refer the reader to [10]. The following result holds.

Theorem 2.1. *Let $\mathfrak{J}$ be a norm ideal in $B(H)$ and $A \in B(H)$. Then* $\left\|\Delta_{[A]}|B(H)/\mathfrak{J}\right\| \leq 2d(A)$.

Proof. By definition
$\|\Delta_{[A]}|B(H)/\mathfrak{J}\| = \sup\left\{\|\Delta_{[A]}([X])\| : [X] \in B(H)/\mathfrak{J}, \|[X]\| = 1\right\}$, where $[X]$ is the canonical image of X in $B(H)/\mathfrak{J}$.
Now, $\left\|\Delta_{[A]}([X])\right\|_{B(H)/\mathfrak{J}} = \|[A][X] - [X][A]\|_{B(H)/\mathfrak{J}} =$
$\|([A]-\lambda)[X]-[X]([A]-\lambda)\|_{B(H/\mathfrak{J}} \leq \|([A]-\lambda)[X]\|_{B(H)/\mathfrak{J}}+\|[X]([A]-\lambda)\|_{B(H)/\mathfrak{J}} \leq$
$2\,\|[A] - \lambda\|\|[X]\|_{B(H)/\mathfrak{J}}\,,\ \lambda \in \mathbb{C}$.
It therefore follows that

$$\left\|\Delta_{[A]}|B(H)/\mathfrak{J}\right\| \leq 2d([A]) \tag{3}$$

where $d([A]) = \inf_{\lambda\in\mathbb{C}} \|[A] - \lambda\|$. But since the map $B(H) \longrightarrow B(H)/\mathfrak{J}$ is continuous and $\|A + \mathfrak{J}\| \leq \|A\|$, see [7], it follows that $d([A]) = \inf_{\lambda\in\mathbb{C}} \|[A] - \lambda\| = \inf_{\lambda\in\mathbb{C}} \|A + \mathfrak{J} - \lambda\| \leq \inf_{\lambda\in\mathbb{C}} \|A - \lambda\| = d(A)$. Thus, it becomes clear that $d([A]) \leq d(A)$. Hence from equation (3), it follows that $\|\Delta_{[A]}|B(H)/\mathfrak{J}\| \leq 2d(A)$. □

The following corollary is immediate,

Corollary 2.2. *Let $A \in B(H)$ and $\mathfrak{J}$ a norm ideal in $B(H)$, then $\|\Delta_{[A]}|B(H)/\mathfrak{J}\| \leq \|\Delta_A|B(H)\|$.*

Proof. This follows immediately from Theorem 2.1 and the identity (1) due to Stampfli [11]. □

Relating the norm of a derivation on an ideal of an algebra and that on the quotient of the algebra by the same ideal has remained a difficult problem in the past. This is due to the fact that ideals being subspaces of the algebra ranges from the trivial ideal to the whole algebra itself. Surprisingly as we shall realize later in this paper, the concept of S - universal operators enables us to tackle a reasonable part of this problem. The following Lemma provides a clear relationship between the norm of inner derivation on the quotient algebra and on the ideal.

Lemma 2.3. *If $A \in B(H)$ is S - universal, and $\mathfrak{J}$ a norm ideal in $B(H)$, then $\|\Delta_{[A]}|B(H)/\mathfrak{J}\| \leq \|\Delta_A|\mathfrak{J}\|$.*

Proof. Since A is S - universal, then by Theorem 2.1, we obtain the desired result. □

Remark 2.4. *From Lemma 2.3, the following question seems natural; When does equality $\|\Delta_{[A]}|B(H)/\mathfrak{J}\| = \|\Delta_A|\mathfrak{J}\|$ hold?*

We satisfactorily provide an answer to the question in Remark 2.4 in the following result,

Theorem 2.5. *Let $B(H)$ be the algebra of bounded linear operators on a Hilbert space H, $\mathfrak{J}$ a primitive norm ideal in $B(H)$. Then for an S - universal operator $A \in B(H)$, $\|\Delta_{[A]}|B(H)/\mathfrak{J}\| = \|\Delta_A|\mathfrak{J}\|$.*

Proof. By Lemma 2.3, if $A \in B(H)$ is S - universal, then $\|\Delta_{[A]}|B(H)/\mathfrak{J}\| \leq \|\Delta_A|\mathfrak{J}\|$.
We therefore show the reverse inequality. Since $\|\Delta_A|\mathfrak{J}\| = \sup\{\|\Delta_A(X)\| : X \in \mathfrak{J}, \|X\|_{\mathfrak{J}} = 1\}$, it follows that $\forall \epsilon > 0$, $\exists X \in \mathfrak{J}$ with $\|X\|_{\mathfrak{J}} = 1$ such that $\|\Delta_A|\mathfrak{J}\| < \|\Delta_A(X)\|_{\mathfrak{J}} + \epsilon$. Also for all primitive ideals $\mathfrak{J}$ in a C* - algebra $\mathfrak{A}$, there exists $\epsilon > 0$ such that $\left\|\Delta_{[A]}([X])\right\|_{\mathfrak{A}/\mathfrak{J}} > \|\Delta_A(X)\|_{\mathfrak{J}} - \epsilon$, see [1]. So by taking $\mathfrak{A} = B(H)$, we have $\|\Delta_A|\mathfrak{J}\| < \|\Delta_A(X)\|_{\mathfrak{J}} + \epsilon < \left\|\Delta_{[A]}([X])\right\|_{B(H)/\mathfrak{J}} + 2\epsilon \leq \left\|\Delta_{[A]}|B(H)/\mathfrak{J}\right\| + 2\epsilon$.
Since ϵ was picked arbitrarily, it follows that $\|\Delta_A|\mathfrak{J}\| \leq \left\|\Delta_{[A]}|B(H)/\mathfrak{J}\right\|$ which completes our proof. □

3 Hyponormal Operators

In this section, we will be particularly interested in operators belonging to the Hilbert - Schmidt class $C_2(H)$. Interestingly, the class $C_2(H)$ together with a norm $\|.\|_2$ defined on it form a norm ideal in $B(H)$. Now, setting $\mathfrak{J} = C_2(H)$, we shall have $(\mathfrak{J}, \|.\|_{\mathfrak{J}})$ as $(C_2, \|.\|_2)$.
Having introduced the notion of S - universal operator in [4], the author showed that a subnormal operator is S - universal if and only if the diameter of the spectrum is equal to twice the radius of the smallest disc containing it. He left open the case of a hyponormal operator. In [2], it is proved that the same conclusion holds true for an arbitrary hyponormal operator. In establishing this result in [2], the authors used the following theorem, see [9]:

Theorem 3.1. *For every hyponormal operator A on a Hilbert space H, there exists a normal operator N and a unitary operator U on some Hilbert space K, and a contraction R of H into K, such that;*

(a) $A = R^*NR$

(b) $\|N\| = \|A\|$

(c) $NU = UN = N^*$

(d) $\|R^*Ug\| \leq \|R^*g\|$ *for all* $g \in K$

(e) *The manifolds $L_n = U^nRH$ (n=0,1,...) form a non- decreasing sequence and span K*

(f) *For any complex scalars* α, β,

$$\sigma(\alpha N + \beta N^*) \subseteq \sigma_l(\alpha A + \beta A^*)$$

(σ_l; left spectrum).

In concluding their paper [2], the authors wondered whether the equality $\|\Delta_N|C_2\| = \|\Delta_A|C_2\|$ holds.
Our result in this section is the following,

Theorem 3.2. *Let A and N be defined as in Theorem 3.1 above. If A is S - universal, then* $\|\Delta_N|C_2\| = \|\Delta_A|C_2\|$.

Proof. Since A is S - universal, then by assertion (b) of Theorem 3.1 and Corollary 3 of [2], it follows that $\|\Delta_A|C_2\| = \|\Delta_A|B(H)\| = 2d(A) = 2\|A\| = 2\|N\| = 2d(N) = \|\Delta_N|B(K)\|$. Then we have

$$\begin{aligned} \|\Delta_A|C_2\| = 2\|A\| &\iff \sup\{\|\Delta_A(X)\| : X \in C_2,\ \|X\|_2 = 1\} = 2\|A\| \\ &\iff \exists\{X_n\} \in C_2(H) \text{ with } \|X_n\|_2 = 1 \text{ such that} \\ &\qquad \|AX_n - X_nA\|_2 \longrightarrow 2\|A\| \text{ as } n \longrightarrow \infty. \end{aligned}$$

Since $\|AX_n - X_nA\|_2 \le \|AX_n\|_2 + \|X_nA\|_2 \le \|A\| + \|X_nA\|_2 \le 2\|A\|$, we deduce that $\|AX_n\|_2 \longrightarrow \|A\|$ as $n \longrightarrow \infty$.
Similarly, we get $\|X_nA\|_2 \longrightarrow \|A\|$ as $n \longrightarrow \infty$.
Now, from the identity

$$\|AX_n - X_nA\|_2^2 = \|AX_n\|_2^2 + \|X_nA\|_2^2 - 2\mathfrak{R}(\langle AX_n, X_nA\rangle),$$

we conclude that $-\mathfrak{R}(\langle AX_n, X_nA\rangle) \longrightarrow \|A\|^2$ as $n \longrightarrow \infty$, where $\mathfrak{R}$ denotes the real part.
But for every sequence $\{X_n\} \in C_2(H)$, there exists a corresponding sequence $\{RX_nR^*\} \in C_2(K)$ such that $\|RX_nR^*\| \le 1$. Moreover

$$\begin{aligned} \langle NRX_nR^*, RX_nR^*N\rangle &= tr((NRX_nR^*)(RX_nR^*N)^*) \\ &= \langle AX_n, X_nA\rangle. \end{aligned}$$

Hence, $\mathfrak{R}(\langle NRX_nR^*, RX_nR^*N\rangle) = \mathfrak{R}(\langle AX_n, X_nA\rangle) \longrightarrow -\|A\|^2 = -\|N\|^2$ as $n \longrightarrow \infty$.
Also

$$\begin{aligned} |\mathfrak{R}(\langle NRX_nR^*, RX_nR^*N\rangle)| &\le \|NRX_nR^*\|_2\|RX_nR^*N\|_2 \\ &\le \|N\|^2. \end{aligned}$$

So it follows that $\|NRX_nR^*\|_2 \longrightarrow \|N\|$ and $\|RX_nR^*N\|_2 \longrightarrow \|N\|$ as $n \longrightarrow \infty$.
Therefore

$$\begin{aligned} \|N(RX_nR^*) - (RX_nR^*)N\|_2^2 &= \|NRX_nR^*\|_2^2 + \|RX_nR^*N\|_2^2 \\ &\quad -2\Re(\langle NRX_nR^*, RX_nR^*N\rangle) \\ &\longrightarrow \|N\|^2 + \|N\|^2 - (-2\|N\|^2) \\ &= 4\|N\|^2 \text{ as } n \longrightarrow \infty. \end{aligned}$$

In other words,

$$\|\Delta_N(RX_nR^*)\|_2^2 \longrightarrow 4\|N\|^2 \ \textit{as } n \longrightarrow \infty.$$

Implying that $\|\Delta_N(RX_nR^*)\|_2 \longrightarrow 2\|N\|$ *as* $n \longrightarrow \infty$. Since $\|\Delta_N|C_2\| \geq \|\Delta_N(RX_nR^*)\|_2$, it follows that $\|\Delta_N|C_2\| \geq 2\|N\|$. But we have that $\|\Delta_N|C_2\| \leq 2\|N\|$, so that $\|\Delta_N|C_2\| = 2\|N\|$. Thus by taking into consideration our assumptions at the start of this proof, it immediately follows that $\|\Delta_N|C_2\| = \|\Delta_A|C_2\|$ which completes this proof. □

Remark 3.3. *Theorem 3.2 answers partly the question posed in [2].*

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