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Norms of Inner Derivations on Norm ideals

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Abstract

Let $B(H)$ be the algebra of bounded linear operators on a Hilbert space H and $\mathfrak{J}$ be a norm ideal in $B(H)$. We investigate the relationship between the diameter of the numerical range of an operator $A \in B(H)$ and the norm of inner derivation implemented by A on a norm ideal $\mathfrak{J}$. Further, we consider the applications of S - universality to the above relationship.

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1 Introduction

For $A \in B(H)$, the inner derivation induced by A is the operator Δ_A defined on $B(H)$ by $\Delta_A(X) = AX - XA,\ \ X \in B(H)$. The norm of an inner derivation Δ_A on H has been computed by J. G. Stampfli [3] as;

$$\|\Delta_A|B(H)\| = 2d(A) \tag{1}$$

where $d(A) = \inf\{\|A - \lambda\| : \lambda \in \mathbb{C}\}$.
In fact, for any normed algebra, each inner derivation is bounded and $\|\Delta_A\| \leq 2\|A\|$, see [2]. We define multiplication operators L_A and R_A, respectively on

$B(H)$ by $L_A(X) = AX$ and $R_A(X) = XA$, for all $X \in B(H)$. These are called left and right multiplication by A respectively.
The numerical range of $A \in B(H)$ is defined by $W(A) = \{\langle Ax, x\rangle : x \in H, \|x\| = 1\}$ and the numerical radius of A is defined by $\omega(A) = \sup\{|\lambda| : \lambda \in W(A)\}$. The spectrum of an operator A, $\sigma(A)$, consists of those complex numbers λ such that $A - \lambda I$ is not invertible while the spectral radius is given by $r(A) = \sup\{|\lambda| : \lambda \in \sigma(A)\}$.
The relationship between the numerical range and spectra has been studied by several mathematicians, see for instance [4], [8]. Recall that approximate point spectrum of A, $\sigma_{ap}(A)$, consists of those complex numbers λ for which there exists a unit sequence $\{x_n\}_n \subseteq H$ such that $\lim_n \|(A-\lambda)x_n\| = 0$. Since the boundary of $\sigma(A)$ is contained in the $\sigma_{ap}(A)$ [8], $\|A\| \in \sigma(A)$ if and only if $\|A\| \in \sigma_{ap}(A)$. We also have that $\sigma(A) \subseteq \overline{W(A)}$ (spectral inclusion) see [4], where the bar denotes the closure. It also turns out in [4] that if $\omega(A) = \|A\|$, then $r(A) = \|A\|$. This result together with the spectral inclusion implies that the norm $\|A\| \in \overline{W(A)}$ if and only if $\|A\| \in \sigma(A)$.
Section 2 is devoted to the study of inner derivation on norm ideals, where we establish the inequality between the diameter of the numerical range and the norm of inner derivation implemented by $A \in B(H)$ on a norm ideal $\mathfrak{J}$, while in section 3, we apply the concept of S - universality to the theory of inner derivations on norm ideals.

2 Inner derivation on norm ideals

A (symmetric) norm ideal $(\mathfrak{J}, \|.\|_{\mathfrak{J}})$ in $B(H)$ consists of a proper two-sided ideal $\mathfrak{J}$ together with a norm $\|.\|_{\mathfrak{J}}$ satisfying the following conditions;

- $(\mathfrak{J}, \|.\|_{\mathfrak{J}})$ is a Banach space.
- $\|AXB\|_{\mathfrak{J}} \leq \|A\|\|X\|_{\mathfrak{J}}\|B\|$, for all $X \in \mathfrak{J}$ and all operators A and B in $B(H)$
- $\|X\|_{\mathfrak{J}} = \|X\|$, for X a rank one operator.

For a complete account of the theory of norm ideals, we refer to [9]. Let $(\mathfrak{J}, \|.\|_{\mathfrak{J}})$ be a norm ideal and let $A \in B(H)$. If $X \in \mathfrak{J}$, then $\Delta_A(X) \in \mathfrak{J}$ and $\|AX - XA\|_{\mathfrak{J}} = \|(A-\lambda)X - X(A-\lambda)\|_{\mathfrak{J}} \leq 2\|A-\lambda\|\|X\|_{\mathfrak{J}}$ for all $\lambda \in \mathbb{C}$. Hence $\|\Delta_A(X)\|_{\mathfrak{J}} \leq 2d(A)\|X\|_{\mathfrak{J}}$, implying that $\|\Delta_A|\mathfrak{J}\| \leq 2d(A)$.

Definition 2.1. *An operator $A \in B(H)$ is S - universal if $\|\Delta_A|\mathfrak{J}\| = 2d(A)$ for each norm ideal $\mathfrak{J}$ in $B(H)$.*

Let K be a non - empty bounded subset of the plane. The diameter of K is defined by $\operatorname{diam}(K) = \sup\{|\alpha - \beta| : \alpha, \beta \in K\}$.

Lemma 2.2. *Let $A \in B(H)$ be non - zero and $\mathfrak{J}$ be an ideal in $B(H)$. If $B \in \mathfrak{J}$ with $Bx_n = y_n$ and $x_n, y_n \in H$ such that $\|y_n\| = \|x_n\| = 1 \, \forall \, n$, then B is unitary.*

Proof. $\langle B^*Bx_n, x_n\rangle = \langle Bx_n, Bx_n\rangle = \|Bx_n\|^2 = \|y_n\|^2 = \|x_n\|^2 = \langle x_n, x_n\rangle = \langle Ix_n, x_n\rangle$. So that $B^*B = I$. Similarly, it is easy to show that $BB^* = I$ which completes the proof. □

We proceed to give an alternative proof to the following result due to L. Fialkow [5].

Theorem 2.3. *For any operator $A \in B(H)$ and each norm ideal $\mathfrak{J}$ in $B(H)$,* $\mathrm{diam}(W(A)) \leq \|\Delta_A|\mathfrak{J}\|$.

Proof. Since $\|\Delta_A|\mathfrak{J}\| = \sup\{\|\Delta_A(B)\| : B \in \mathfrak{J}, \|B\| = 1\}$, then $\|\Delta_A|\mathfrak{J}\| \geq \|AB - BA\|$ for all $B \in \mathfrak{J}$ with $\|B\| = 1$. Hence $\exists \, \{x_n\} \subset H$ with $\|x_n\| = 1 \; \forall \; n$ such that $\|AB - BA\| \geq \|ABx_n - BAx_n\| \geq \|ABx_n\| - \|BAx_n\|$.
But since $|\langle ABx_n, Bx_n\rangle| \leq \|ABx_n\|\|Bx_n\| \leq \|ABx_n\|\|B\|\|x_n\| = \|ABx_n\|$ and $|\langle BAx_n, Bx_n\rangle| \leq \|BAx_n\|$, we have $\|\Delta_A|\mathfrak{J}\| \geq \|ABx_n\| - \|BAx_n\| \geq |\langle ABx_n, Bx_n\rangle| - |\langle BAx_n, Bx_n\rangle|$. But $Bx_n = y_n$ with $\|y_n\| = \|x_n\| = 1$. So by Lemma 2.2, $\langle ABx_n, Bx_n\rangle = \langle Ay_n, y_n\rangle$ and $\langle BAx_n, Bx_n\rangle = \langle Ax_n, B^*Bx_n\rangle = \langle Ax_n, x_n\rangle$.
Thus from [8], it follows that $\|\Delta_A|\mathfrak{J}\| \geq |\langle Ay_n, y_n\rangle - \langle Ax_n, x_n\rangle| = \{|\alpha - \beta|; \alpha, \beta \in W(A)\}$. This implies that $\|\Delta_A|\mathfrak{J}\| \geq \sup\{|\alpha - \beta| : \alpha, \beta \in W(A)\}$.
Hence $\|\Delta_A|\mathfrak{J}\| \geq \mathrm{diam}(W(A))$. □

Remark 2.4. *The following question seems natural; When does equality $\|\Delta_A|\mathfrak{J}\| = diam(W(A))$ hold?*

3 Applications to S - universality

The notion of S - universal operators was introduced by L. Fialkow in order to effectively study the extent to which the identity (1) applies, see [5]. Before stating our results in this section, we need some additional preliminaries. We begin by noting that for any operator $A \in B(H)$, the inner derivation Δ_A can also be represented as $\Delta_A = L_A - R_A$.
Let $C_p(H)$ denote the Schatten p - ideal, $1 \leq p \leq \infty$, see for instance [9]. The space $C_p(H)$ consists of the compact operators X such that $\sum_j S_j^p(X) < \infty$, where $\{S_j(X)\}_j$ denotes the sequence of singular values of X. For $X \in C_p(H)$ $(1 \leq p \leq \infty)$, we set $\|X\|_p = (\sum_j S_j^p(X))^{\frac{1}{p}}$, where, by convention, $\|X\|_\infty = S_1(X)$ is the usual operator norm of X. Then $(C_p(H), \|.\|_p)$ is a norm ideal. Moreover, $(C_2(H), \|.\|_2)$ is a Hilbert space with inner product defined by $\langle X, Y\rangle = tr(XY^*)$, $(X, Y \in C_2(H))$, where tr stands for the usual

trace functional and Y^* denotes the adjoint of Y. We write $\Delta_A|C_2$ instead of $\Delta_A|C_2(H)$ to denote inner derivation on $C_2(H)$. We shall be interested in operators belonging to $C_2(H)$.
The theory of the numerical range and the spectra of inner derivations on norm ideals was studied by several mathematicians, see for instance [5] or [10]. In [10], S. Shaw considered inner derivations acting on subspaces which satisfy axioms like those of norm ideals. In particular, he proved that $\overline{W(\Delta_A|C_2)} = \overline{W(A)} - \overline{W(A)}$. This formed the numerical range analogue of Fialkow's [5] formula for spectra, which states that $\sigma(\Delta_A|C_2) = \sigma(A) - \sigma(A)$. Fialkow's work [5] followed from the work of A. Brown and C. Pearcy [1] who studied the multiplication operators L_A and R_A and established that $\sigma(L_A) = \sigma(R_A) = \sigma(A)$.
The following result is due to Barraa and Boumazgour,

Theorem 3.1. *Let A, $B \in B(H)$ be non - zero. Then the equation $\|A + B\| = \|A\| + \|B\|$ holds if and only if $\|A\|\|B\| \in \overline{W(A^*B)}$*

See [7] for the proof.
The following result will hold,

Theorem 3.2. *Let $A \in B(H)$ be S - universal. Then*

$$diam(W(A)) = 2\|A\|.$$

Proof. Since A is S - universal, then $\|\Delta_A|C_2\| = 2d(A)$. But by Stampfli [3], for any $A \in B(H)$, $\|\Delta_A|B(H)\| = 2d(A) = 2\inf_{\lambda\in\mathbb{C}} \|A - \lambda\|$ and by compactness, $\exists\ \mu \in \mathbb{C}$ such that $\inf_{\lambda\in\mathbb{C}} \|A - \lambda\| = \|A - \mu\|$ ([6]). Hence $\|\Delta_A|C_2\| = 2\|A - \mu\|$. Since $\Delta_A|C_2 = \Delta_{A-\mu}|C_2 = L_{A-\mu}|C_2 - R_{A-\mu}|C_2$, it follows that $\|L_{A-\mu}|C_2 - R_{A-\mu}|C_2\| = 2\|A - \mu\|$. On the other hand, since $\|L_{A-\mu}|C_2\| = \|A - \mu\|$ and $\|R_{A-\mu}|C_2\| = \|A - \mu\|$, it follows that $\|L_{A-\mu}|C_2 - R_{A-\mu}|C_2\| = \|L_{A-\mu}|C_2\| + \|R_{A-\mu}|C_2\|$. Without loss of generality, we may assume that $\mu = 0$ so that $\|L_A|C_2 - R_A|C_2\| = \|L_A|C_2\| + \|R_A|C_2\|$. Then by Theorem 3.1, this is equivalent to

$$\|L_A|C_2\|\|R_A|C_2\| \in \overline{W(-L_{A^*}|C_2R_A|C_2)}.$$

As remarked in the introduction this implies that

$$\|L_A|C_2\|\|R_A|C_2\| \in \sigma(-L_{A^*}|C_2R_A|C_2).$$

But $\sigma(-L_{A^*}|C_2R_A|C_2) = -\sigma(A^*)\sigma(A)$ and $\|A\|^2 = \|L_{A^*}|C_2\|\|R_A|C_2\|$. So $\exists\ \alpha, \beta \in \sigma(A)$ such that $\|A\|^2 = -\overline{\alpha}\beta$.
Since $|\alpha| \leq \|A\|$ and $|\beta| \leq \|A\|$, one can find $\theta \in \mathbb{R}$ such that $\alpha = \|A\|e^{i\theta}$ and $\beta = -\|A\|e^{i\theta}$. Also since $\sigma(\Delta_A|C_2) = \sigma(A) - \sigma(A)$, it follows that $r(\Delta_A|C_2) =$

diam($\sigma(A)$). So that $r(\Delta_A|C_2) = \text{diam}(\sigma(A)) \geq |\alpha - \beta| = 2\|A\|$. By the spectral inclusion, $\sigma(A) \subseteq \overline{W(A)}$ [8], it follows that $\text{diam}(\sigma(A)) \leq \text{diam}(W(A))$ and so $\text{diam}(W(A)) \geq \text{diam}(\sigma(A)) \geq 2\|A\|$, that is

$$\text{diam}(W(A)) \geq 2\|A\|. \tag{2}$$

Conversely, we need to establish the reverse inequality. By definition $\text{diam}(W(A)) = \sup\{|\alpha - \beta| : \alpha, \beta \in W(A)\}$. This implies that $\exists\ x, y \in H$ with $\|x\| = \|y\| = 1$ such that $\alpha = \langle Ax, x\rangle$ and $\beta = \langle Ay, y\rangle$. So that $|\alpha - \beta| = |\langle Ax, x\rangle - \langle Ay, y\rangle| \leq |\langle Ax, x\rangle| + |\langle Ay, y\rangle| \leq 2\|A\|$.
Thus

$$\text{diam}(W(A)) \leq 2\|A\|. \tag{3}$$

Now, from equations (3) and (2), we obtain our result. □

The following Theorem will therefore answer the question in Remark 2.4 above.

Theorem 3.3. *Let $A \in B(H)$ be S - universal and $\mathfrak{J}$ a norm ideal in B(H). Then diam$(W(A)) = \|\Delta_A|\mathfrak{J}\|$.*

Proof. From Theorem 2.3, it turns out that for any $A \in B(H)$, $\text{diam}(W(A)) \leq \|\Delta_A|\mathfrak{J}\|$. Our task therefore is to establish the reverse inequality, that is, $\text{diam}(W(A)) \geq \|\Delta_A|\mathfrak{J}\|$. Now, since A is S - universal, then by Theorem 3.2 above, we have $\text{diam}(W(A)) = 2\|A\|$ and $\|\Delta_A|B(H)\| = 2d(A) = \|\Delta_A|\mathfrak{J}\|$, see [3]. But $B(H)$ being a normed algebra, it follows that $\|\Delta_A|\mathfrak{J}\| = \|\Delta_A|B(H)\| \leq 2\|A\| = \text{diam}(W(A))$. Hence $\text{diam}(W(A)) \geq \|\Delta_A|\mathfrak{J}\|$. This completes the proof. □

The following Corollaries are immediate,

Corollary 3.4. *If $A \in B(H)$ is S - universal, then diam$(W(A)) = \|\Delta_A|B(H)\|$.*

Proof. This follows immediately from Theorem 3.3 and the definition of an S - universal operator. □

Corollary 3.5. *If $A \in B(H)$ is S - universal, then $\|\Delta_A|B(H)\| = 2\|A\|$.*

Proof. The proof of this Corollary follows immediately from Theorem 3.3 and the Corollary 3.4 above. □

The next result considers the Hilbert - Schmidt class $C_2(H)$ and establishes the necessary and sufficient condition for a non - zero operator $A \in B(H)$ to be S - universal.
The following results hold,

Theorem 3.6. *Let $A \in B(H)$ be non - zero. Then $\|\Delta_A|C_2\| = \|\Delta_A|B(H)\|$ if and only if $r(\Delta_A|C_2) = \|\Delta_A|B(H)\|$*

Corollary 3.7. *For $A \in B(H)$, the following are equivalent:*

1. *A is S - universal*
2. *diam$(W(A)) = 2\inf_{\lambda\in\mathbb{C}} \|A-\lambda\|$*
3. *diam$(\sigma(A)) = 2\inf_{\lambda\in\mathbb{C}} \|A-\lambda\|$*

The proofs of the above results can be found in [7].
The necessary and sufficient condition for A to be S - universal follows below,

Theorem 3.8. *Let $A \in B(H)$ be non- zero. Then A is S - universal if and only if $r(\Delta_A|C_2) = \omega(\Delta_A|C_2)$.*

Proof. We first assume that A is S - universal. Since $\overline{W(\Delta_A|C_2)} = \overline{W(A)} - \overline{W(A)}$, it follows that $\omega(\Delta_A|C_2) = \mathrm{diam}(W(A))$. And because $\sigma(\Delta_A|C_2) = \sigma(A) - \sigma(A)$, we have $r(\Delta_A|C_2) = \mathrm{diam}(\sigma(A))$. Now A is S - universal, so by corollary 3.4 and definition of an S - universal operator, we have $\mathrm{diam}(W(A)) = \|\Delta_A|B(H)\|$ and $\|\Delta_A|B(H)\| = \|\Delta_A|C_2\|$, respectively. Thus

$$\omega(\Delta_A|C_2) = \mathrm{diam}(W(A)) = \|\Delta_A|B(H)\| = \|\Delta_A|C_2\|. \quad (4)$$

By Theorem 3.6 due to Barraa and Boumazgour [7], equation (4) becomes

$$\omega(\Delta_A|C_2) = \|\Delta_A|B(H)\| = r(\Delta_A|C_2). \quad (5)$$

That is $r(\Delta_A|C_2) = \omega(\Delta_A|C_2)$.
Conversely, assume that $r(\Delta_A|C_2) = \omega(\Delta_A|C_2)$. Then since $\omega(\Delta_A|C_2) = \mathrm{diam}(W(A))$ and $r(\Delta_A|C_2) = \mathrm{diam}(\sigma(A))$, we have $\mathrm{diam}(W(A)) = \mathrm{diam}(\sigma(A))$. Thus, by corollary 3.7 above, it follows immediately that A is S - universal. □

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