On the Numerical Range of
a Generalized Derivation

F. M. Runji

Department of Mathematics Statistics and Actuarial Science
Karatina University, Karatina, Kenya

J. O. Agure

Department of Pure and Applied Mathematics
Maseno University, Maseno, Kenya

F. O. Nyamwala

Department of Physics and Mathematics
Moi University, Eldoret, Kenya

Abstract

We examine the relationship between the numerical range of the restriction of a generalized derivation to a norm ideal J and that of its implementing elements.

1 Introduction

Given a Banach algebra $\mathscr{A}$, $\mathscr{A}^*$ the dual of $\mathscr{A}$, $S(\mathscr{A}) = \{x \in \mathscr{A} : \|x\| = 1\}$, the unit sphere, and $x \in S(\mathscr{A})$, let $D(x, \mathscr{A}) = \{f \in \mathscr{A}^* : f(x) = 1 = \|f\|\}$.

The Hahn-Banach theorem guarantees that $D(x, \mathscr{A})$ is non empty for each $x \in S(\mathscr{A})$. The elements of $D(I, \mathscr{A})$, I, the identity in $\mathscr{A}$, are called normalized states or simply states.

For $a \in \mathscr{A}$, and $x \in S(\mathscr{A})$, we define $V(x, a, \mathscr{A}) = \{f(ax) : f \in D(x, \mathscr{A})\}$.

The numerical range of a is the set $V(a, \mathscr{A}) = \bigcup \{V(x, a, \mathscr{A}) : x \in S(\mathscr{A})\}$.

Given a Banach space $\mathscr{H}$, we may consider the Banach algebra $\mathscr{A} = L(\mathscr{H})$ and define define the spatial numerical range of A by

$W(A; L(\mathscr{H})) = \{f(Ax) : f \in \mathscr{H}^*, x \in \mathscr{H}, and \|f\| = \|x\| = 1 = f(x)\}$

We first give some basic properties of the numerical range .

Bonsal [4], has shown that $V(a, \mathscr{A}) = V(I, a, \mathscr{A})$, and for each $a \in \mathscr{A}, V(a, \mathscr{A})$ is a compact convex subset of $\mathbb{C}$.

Lemma 1. $V(x, a, \mathscr{A}) = \{f(ax) : f \in D(x, \mathscr{A})\}$ *is convex.*

Proof. Let $\lambda_1, \lambda_2 \in V(x, a, \mathscr{A})$. Then there exist support functionals $f_1, f_2 \in D(x, \mathscr{A})$ such that $\lambda_1 = f_1(ax)$,$\lambda_2 = f_2(ax)$.

Define f on $D(x, \mathscr{A})$ by $f(ax) = tf_1(ax) + (1-t)f_2(ax), t \in (0, 1)$. We need to show that $f \in D(I, \mathscr{A})$ Clearly f is linear and $|f(ax)| = |tf_1(ax) + (1 - t)f_2(ax)| \leq t |f_1(ax)| + (1-t) |f_2(ax)| \leq t \|f_1\| \|ax\| + (1-t) \|f_2\| \|ax\| = \|ax\| \Rightarrow \|f\| \leq 1$.

Also, $f(x) = tf_1(x) + (1 - t)f_2(x) = 1$

$\Rightarrow \|f\| \geq 1$

Thus $f \in D(I, \mathscr{A})$ which is convex and hence $V(x, a, \mathscr{A})$ is convex. $\qquad \square$

For $a \in \mathscr{A}$, we define the left multiplication operator $L_a : \mathscr{A} \to \mathscr{A}$ by $L_a(x) = ax, \forall x \in \mathscr{A}$ and $\|L_a\| = sup \{\|ax\| : x \in \mathscr{A}, \|x\| \leq 1\}$

L_a is a linear operator in $\mathscr{A}$ and also a bounded operator since

$\|L_a\| = sup \{\|ax\| : x \in \mathscr{A}, \|x\| \leq 1\} \leq \|a\|$.

$L_a(\mathscr{A})$ will denote the set of all left multiplication operators on the algebra $\mathscr{A}$ as a ranges on $\mathscr{A}$ .This set is a normed algebra.

The algebraic numerical range of $L_a \in L_a(\mathscr{A})$ is the non-empty set:

$V(L_a; L_a(\mathscr{A})) = \{f(L_a); f \in L_a(\mathscr{A})^*, f(L_e) = 1 = \|f\|\}$.

Similarly the right multiplication operator for $b \in \mathscr{A}$ is defined as ;

$R_b : \mathscr{A} \to \mathscr{A}, x \to xb$

We note that $\forall x \in \mathscr{A}$ and fixed $a, b \in \mathscr{A}$, $\Delta_{a,b}(x) = L_a(x) - R_b(x) = ax - xb$, is the generalized derivation induced by $a, b \in \mathscr{A}$.

In [3], it is shown that for any Banach algebra $\mathscr{A}$, $\|L_a\| = \|a\| = \|R_a\|$ and that $V(a; \mathscr{A}) = V(L_a; L(\mathscr{A})) = V(R_a; L(\mathscr{A}))$, $L(\mathscr{A})$ the algebra of the bounded linear operators on $\mathscr{A}$.

Lemma 2. *For $a \in \mathscr{A}, L_a \in L_a(\mathscr{A}), \|L_a\| = \|a\| = \|R_a\|$*

Proof.

$$\|L_a\| = sup\,\{\|L_a(x)\| : \|x\| = 1\}$$
$$= sup\,\{\|ax\| : \|x\| = 1\}$$
$$\leq \|a\|\,\|x\|$$
$$\Rightarrow \|L_a\| \leq \|a\| \quad\text{........................}(i)$$

If $\mathscr{A}$ has unit e, we have $L_a(e) = ae = a$ which implies $\|a\| = \|L_a(e)\| \leq$
$\|L_a\|\,\|e\| = \|L_a\| \Rightarrow \|L_a\| \geq \|a\| \quad\text{.......................}(ii)$
From (i) and (ii) equality follows.
Similarly we obtain $\|R_a\| = \|a\|$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Lemma 3. *For $a \in \mathscr{A}, V(a;\mathscr{A}) = V(L_a; L(\mathscr{A})) = V(R_a; L(\mathscr{A}))$*

Proof. Let $\lambda \in V(a : \mathscr{A})$, Then there exist $f \in S(\mathscr{A})$ such that $f(a) = \lambda$
Now define F on $L(\mathscr{A})$ by
$F(L_a) = f(ax)$, for all $L_a \in L(\mathscr{A})$.
Clearly F is linear since

$$F(\alpha L_a + \beta L_b) = f(\alpha ax + \beta bx)$$
$$= f(\alpha ax) + f(\beta bx)$$
$$= \alpha f(ax) + \beta f(bx)$$
$$= \alpha F(L_a) + \beta F(L_b), a, b \in \mathscr{A}, \alpha, \beta \in \mathbb{C}$$

f is also bounded and positive since
$\|F(L_a)\| = sup\,\{\|f(ax)\|\} \leq \|f\|\,\|ax\| = c\,\|L_a\|$.
Also $F(L_e) = f(ex) = f(x) = 1$ and $\|F\| = 1$.
So F as defined is a positive linear functional on $\mathscr{A}$.
Take a finite rank operator $b \in L(\mathscr{A})$ defined by
$bx = g(x)a$, for all $x \in \mathscr{A}, g \in S(\mathscr{A})$. Clearly $\|b\| = 1$ and $F(b) = f(bx) = f(g(x)a) = g(x)f(a) = \lambda$. Hence $V(a;\mathscr{A}) \subseteq V(L_a; L(\mathscr{A}))$
Conversely we show that $V(L_a; L(\mathscr{A})) \subseteq V(a;\mathscr{A})$
Let $\lambda \in V(L_a; L(\mathscr{A}))$. Then there exists a state $f \in L(\mathscr{A})^*$ such that $f(L_a) = \lambda$
Define a functional $h \in \mathscr{A}^*$ by $h(a) = f(L_a)$. Then :

$$h(\alpha a + \beta b) = f(\alpha L_a + \beta L_b)$$
$$= f(\alpha L_a) + f(\beta L_b)$$
$$= \alpha f(L_a) + \beta f(L_b)$$
$$= \alpha h(a) + \beta h(b)$$

$\Rightarrow h$ is linear and bounded. h is also positive since $h(a^*a) = f(L_a^* L_a) \geq 0$
Furthermore h is of norm 1 since $h(e) = f(L_e) = 1$ and
$1 = |h(e)| \leq \|h\| \|e\| \Rightarrow \|h\| \geq 1$. We also have

$$\begin{aligned} \|h\| &= \sup\{|h(a)| : \|a\| = 1\} \\ &= \sup\{|f(L_a)| : \|L_a| = 1\} \\ &\leq \|f\| \\ &= 1 \end{aligned}$$

Thus h is a state on $\mathscr{A}^*$ and so $V(L_a; L(\mathscr{A})) \subseteq V(a; \mathscr{A})$ $\square$

2   NORM IDEALS

Let X and Y be Banach algebras. $L(X)$ and $L(Y)$, the algebra of all bounded linear operators on X and Y respectively.
Let $(J, \|.\|_J)$ be a norm ideal on $L(Y, X)$, the algebra of all bounded linear operator from Y to X such that:

i) $(J, \|.\|_J)$ is a Banach space

ii) If $A \in L(X), T \in J, B \in L(Y)$ then $ATB \in J$,and $\|ATB\|_J \leq \|A\| \|T\|_J \|B\|$

iii) $\|T\| \leq \|T\|_J$, $T \in J$and

iv) $\|T\|_J = \|T\|$, for T a rank- one operator.

If $A \in L(X), B \in L(Y)$ and $T \in J$, then the operators L_A, R_B and $L_A - R_B$ are all bounded linear operators on $L(J)$, the space of all bounded linear operators from J to J, where:
$L_A T = AT$, the left multiplication operator,
$R_B T = TB$, the right multiplication operator and
$(L_A - R_B) T = AT - TB$, the generalized derivation.The following lemma will hold.

Lemma 4. $V(A : L(X)) = V(L_A : L(J))$

Proof. Let $\lambda \in V(A : L(X))$. Then there exist $f \in L(X)^*$ such that
$\lambda = f(A), and, f(I_{L(X)}) = 1 = \|f\|$
Let $\mathscr{A}_0 = \{L_A : A \in L(X), L_A(T) = AT, T \in J\} \subseteq L(J)$.
$\mathscr{A}_0$ is a linear subspace of $L(X)$.
On $\mathscr{A}_0^*$, define a linear functional g such that $g(L_A) = f(A)$. Clearly g as defined is a state and the Hahn-Banach theorem guarantees the existence of its extension on $L(J)$. Hence, $V(A : L(X)) \subseteq V(L_A : L(J))$
$\Leftarrow$: suppose $\lambda \in V(L_A : L(J)$. Then $\exists f \in L(J)^*$ such that $f(L_A) = \lambda$ and

$f(I_{L(J)}) = 1 = \|f\|$.

Define a linear operator h on $L(X)^*$ by $h(A) = f(L_A)$. Then $h(I) = f(I_{L(J)}) = 1$.

h is thus a state on $L(X)^*$ and $V(L_A : L(J)) \subseteq V(A : L(X))$ $\qquad\qquad$ $\square$

3 $\quad$ Norm of L_A and R_B in $(J, \|.\|_J)$

Lemma 5. $\|L_A\|_J = \|A\|$

Proof. Condition (ii) above on the definition of a norm ideal implies that L_A and R_B are bounded linear operators on $(J. \|.\|_J)$ and

$$\begin{aligned}
\|L_A\|_J &= Sup\,\{\|AX\| : \|X\|_J = 1, X \in J\} \\
&\leq \|A\|\,\|X\|_J \\
&= \|A\|
\end{aligned}$$

Condition (iii) implies $\|L_A\|_J \geq \|A\|$.

It therefore follows that $\|L_A\|_J = \|A\|$

Similarly $\|R_B\|_J = \|B\|$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

4 $\quad$ Numerical range of the generalized derivation in the norm ideal J

In the past, generalized derivations , their properties and their restrictions to norm ideals have been investigated by many authors. For example,their spectra have been characterized in [7] and [8]. The famous results on the norms of inner derivation and the generalized derivation as obtained by Stampfli [6] using maximal numerical range have ever since provided a crucial lead in defining of norms of elementary operators.We recall the works of Kyle [9] who examines the relationship between the numerical range of an inner derivation, and that of its implementing element.

In his paper, Magajna [2] gives the essential numerical range of the the generalized derivation defined on the Hilbert-Schmidt class in terms of the numerical and the essential numerical ranges of the implementing operators. Shaw [10] in particular, established that the algebra numerical range of a generalized derivation restricted to a norm ideal J is equal to the difference of the algebra numerical ranges of the implementing operators provided that J contains all finite rank operators and is suitably normed . With slight modification we obtain an alternative proof to Shaw's result.

Lemma 6. *Let J be as defined above. Then for $A \in L(X), B \in L(Y), V(\Delta_{A,B} : L(J)) = V(A : L(X)) - V(B : L(Y))$*

Proof. Let $\lambda \in V\left(\Delta_{A,B} : L(J)\right)$. This implies there exist $f \in L(J)^*$ such that $f(\Delta_{A,B}) = \lambda$ and $f\left(I_{L(J)}\right) = 1 = \|f\|$

Let $\mathscr{A}_0 = \{L_A : A \in L(X), L_A(T) = AT, T \in J\} \subseteq L(J)$ and $\mathscr{A}_1 = \{R_B : B \in L(Y), R_B(T) = TB, T \in J\} \subseteq L(J)$ i.e. the set of the left and right multiplication operators respectively in $L(J)$. These are linear subspaces of $L(X)$ and $L(Y)$ respectively. Let also $S(L(J)) = \left\{f \in L(J)^* : f\left(I_{L(J)}\right) = 1 = \|f\|\right\}$

$$\begin{aligned}
\lambda = f\left(\Delta_{A,B} : L(J)\right) &= \{f\left(L_A - R_B : f \in S(L(J))\right)\} \\
&= \left\{f(L_A) : f \in L(X)^*, f(I_{L(X)}) = 1 = \|f\|\right\} \\
&\quad - \left\{f(R_B) : f \in L(Y)^*, f(I_{L(Y)}) = 1 = \|f\|\right\} \\
&= V\left(L_A : L_A \in L(J)\right) - V\left(R_B : R_B \in L(J)\right) \\
&\in V\left(A : L(X)\right) - V\left(B : L(Y)\right)
\end{aligned}$$

To prove the reverse inclusion, we make use of the spatial numerical range. Choose λ in $W\left(A : L(X)\right)$ and μ in $W\left(B : L(Y)\right)$. Then we can find functionals f and g in $L(X)^*, L(Y)^*$ such that

$\|f\| = \|x\| = f(x) = 1$, with $f(Ax) = \lambda$ and
$\|g\| = \|y\| = g(y) = 1$, with $g(By) = \mu$
Let X be a rank one operator in J such that $Xz = g(z)x, \forall z \in Y$,
Also define F in $L(J)^*$ by $F(T) = f(Ty), \forall T \in L(J)$
Then $F(X) = f(Xy) = fg(y)x = g(y)f(x) = 1$,
$F(I) = f(Iy) = fg(y)x = g(y)f(x) = 1$ and
$|F(T)| \leq \|f\| \|T\|_J \|Y\| = \|T\|_J$
Clearly $\|F\|_J = \|X\|_J = 1$ and $\left(I_{L(J)}, F\right) \in L(J) \times L(J)^*$
Thus,

$$\begin{aligned}
F(\Delta_{A,B}(X)) &= F\left(AX - XB\right) \\
&= f\left(AX - XB\right)y \\
&= f(AXy) - f(XBy) \\
&= f(g(y)Ax) - f(g(By)x) \\
&= f(Ax)g(y) - f(x)g(By) \\
&= \lambda - \mu \\
&\in \{W\left(A : L(X)\right) - W\left(B : L(Y)\right)\}
\end{aligned}$$

Now

$$\begin{aligned}
V\left(\Delta_{A,B}; L(J)\right) &= \overline{co}W\left(\Delta_{A,B}; L(J)\right) \\
&\supseteq \overline{co}\left\{W\left(A; L(X)\right) - W\left(B; L(Y)\right)\right\} \\
&= \overline{co}\left\{W\left(A; L(X)\right)\right\} - \overline{co}\left\{W\left(B; L(Y)\right)\right\} \\
&= V\left(A; L(X)\right) - V\left(B; L(Y)\right)
\end{aligned}$$

Thus $\{V(A; L(X)) - V(B; L(Y))\} \subseteq V(\Delta_{A,B}; L(J))$ $\qquad\qquad$ $\square$

References

[1] A. Seddik, The Numerical Range of Elementary Operators, *Integr. Equ. Oper. Theory*, **43** (2002), 248-252. https://doi.org/10.1007/bf01200256

[2] B. Magajna, On the Essential Numerical Range of a Generalized Derivation, *Proc. of the American Mathematical Society*, **99** (1987), no. 1, 86-92. https://doi.org/10.1090/s0002-9939-1987-0866435-1

[3] F.F. Bonsall and J. Duncan, *Numerical Ranges of Operators on Normed Spaces and of Elements of Normed Algebras*, London Math. Soc. Lecture Note Series 2, Cambridge, 1971. https://doi.org/10.1017/cbo9781107359895

[4] F. F. Bonsall, The Numerical Range of an Element of a Normed Algebra, *Glasgow Mathematical Journal*, **10** (1968), 68-72. https://doi.org/10.1017/s0017089500000562

[5] J. Anderson and C. Foias, Properties which Normal Operators Share with Normal Derivations and Related Operators, *Pacific Journal of Mathematics*, **61** (1975), no. 2, 313-325. https://doi.org/10.2140/pjm.1975.61.313

[6] J.G. Stampfli, The Norm of a Derivation, *Pacific J. Math.*, **33** (1970), 737-747. https://doi.org/10.2140/pjm.1970.33.737

[7] L. A. Fialkow, Elements of spectral theory for generalized derivations, *J. Operator Theory*, **3** (1980), 89-113.

[8] L. A. Fialkow, A note on norm ideals and the operator X - AX - XB, *Israel J. Math.*, **32** (1979), 331-348. https://doi.org/10.1007/bf02760462

[9] J. Kyle, Numerical ranges of derivations, *Proc. Edinburgh Math. Soc.*, **21** (1978), 33-39. https://doi.org/10.1017/s0013091500015856

[10] S.Y. Shaw, On Numerical Ranges of Generalized Derivations and Related Properties, *J. Austral. Math. Soc.*, (Series A), **36** (1984), 134-142. https://doi.org/10.1017/s1446788700027397