Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 14, Number 8 (2018), pp. 1121–1147

http://www.ripublication.com/gjpam.htm

Effects of Structural Viscoelasticity in Non-Divergence free Deformable Porous Media with Incompressible Constituents

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Abstract

In this paper, we investigate the importance of structural viscoelasticity in the mechanical response of deformable porous media with incompressible constituents under sudden changes in the external applied quasi-static loads using mathematical analysis. Here, the applied load is characterized by a step pulse or a trapezoidal pulse. Mathematical models of non-divergence free deformable porous media are often used to characterize the behaviour of biological tissues such as cartilages, engineered tissue scaffolds, and bones which are viscoelastic and incompressible in nature, and viscoelasticity may change with age, disease or by design. The problem is formulated as a mixed boundary value problem of the theory of poro-viscoelasticity, in which an explicit solution is obtained in one-dimension. Further, dimensional analysis is utilized to identify dimensionless parameters that can aid the design of structural properties so as to ensure that the fluid velocity past the porous medium remains bounded below a given threshold to prevent potential damage.

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Our investigation shows that the fluid dynamics within the medium could abruptly be altered if the applied load encounters a sudden change in time and structural viscoelasticity is too small. This explains the confined compression experiment clarifying the cause of micro-structural damages in biological tissues associated with loss of tissue viscoelastic property, which leads to the cause of diseases like Glaucoma.

AMS subject classification:
Keywords: Viscoelasticity, Poroviscoelasticity, Deformable porous medium, Flow through porous media, Confined compression, Boundary value problem, Quasi-static loading.

Nomenclature
σ-Effective stress in one dimension (Pa)
σ_e-Elastic stress contribution in one dimension (Pa)
σ_v- Viscoelastic stress contribution in one dimension(Pa)
ϕ-Volumetric fraction of fluid component or porosity (—)
ζ-Variation of fluid content per unit volume of porous media ($N.m^{-2}$)
v-Discharge velocity ($m.s^{-1}$)
u-Solid displacement (m)
$\vec{B}$-Body force per unit of volume ($N.m^{-3}$)
Q-Net volumetric fluid production rate ($m^3.s^{-1}$)
$\mathbf{K}$-Permeability tensor (m^2)
p-Darcy fluid pressure (Pa)
$\partial\Omega$-Boundary of the domain Ω (—)
$\mathcal{P}$-Fluid power density ($W.m^{-3}$)
L-Length of material (m)
δ-Viscoelastic effects for the solid components (—)
μ_e & λ_e-Lame elastic parameters
μ_v & λ_v-viscoelastic parameters
λ_n-Eigenvalues (m^{-2})
∇u-Volumetric dialation (—)
ρ-Fluid mass density (kgm^{-3})
T-Time (s)
$\mathbf{I}$-Identity tensor (—)

1. Introduction

Fluid flow in deformable porous media has many applications in biology, medicine and bioengineering. For examples, blood flow through tissues in the human body [21, 9, 13, 30], and fluid flow through cartilages, bones and engineered tissue scaffolds which exhibit both elastic and viscoelastic behaviours resulting from the combined action of various components, including elastin, collagen, and extracellular [22], and from mathematical viewpoints, the study of fluid flow through deformable porous structures requires the coupling of poro-elasticity with structural viscoelasticity, leading to Poro-Viscoelastic Models. Interestingly, clinical evidence has shown that the loss of viscoelastic biological tissues properties which leads to their damages has been linked to various diseases conditions such as atherosclerosis, Alzheimer's disease and gluocoma. This triggered the interests of scientists to investigate the cause of the micro-structural damages in the tissues.

The theoretical study of fluid flow through deformable porous media has attracted a lot of attentions since the beginning of the last century. The development of the field started with the work of [27]. But, it was [5] who set up the framework and ignited the mathematical development for fluid flow through poro-elastic media. Biot pioneered the development of an elastodynamics theory for a saturated elastic porous medium. This theory has had wide applications in the geophysics for analysing wave propagation characteristics under cyclic load. To date, several books and articles have been devoted to the mathematical analysis and numerical investigation of poro-elastic models, such as [7], with applications ranging from engineering and geophysics to medicine and biology. [17] devised a theoretical and numerical framework to investigate poro-elastic and poro-viscoelastic models. The study showed that structural viscoelasticity plays an important role in determining the regularity requirements for volumetric and boundary forcing terms, as well as for the corresponding solutions. More so, the solutions of fluid-solid mixture (elastic displacement, fluid pressure, and Darcy velocity) are more sensitive to the boundary traction in the elastic case than in the viscoelastic scenario [3].

Further, the study in [17] has provided numerical clues that sudden changes in body forces and/ or stress boundary conditions may lead to uncontrollable fluid-dynamical responses within the medium in the absence of structural viscoelasticity. The causes of the damage in biological tissues, namely that sudden time variations in stress conditions coupled with lack of structural viscoelasticity could lead to microstructural damage due to local fluid-dynamical alterations [28].

In this paper, we consider a non-linear system of partial differential equations that are often encountered when modeling fluid flow through deformable porous media. The non-linearity in the system is as a result of the medium permeability.

The relevance of the combination between structural mechanics and fluid dynamics in the damage of deformable porous media has been the focus of many authors [18, 24]. In this paper, we investigate a particular aspect of this combination and we focus at characterising and quantifying the role of structural viscoelasticity on the biomechanical response to abrupt changes in stress conditions. Using the fact that biomechanical applications are characterised based on tissues having the same mass density as water, we consider in this paper the situation whereby deformable porous media are composed of incompressible solid and fluid components. In this case, it follows from Biot's theory of poro-elasticity (in $\mathbb{R}^n,\ n = 1, 2, 3$) that increment of fluid content ζ is given by;

$$\zeta = \mathfrak{b}\epsilon(\vec{u}) + s_\epsilon p, \tag{1.1}$$

where $\mathfrak{b}$ is the Biot Willi's constant, $\vec{u}$ is the solid displacement, $\epsilon(\vec{u})$ is the volumetric strain, s_ϵ is storage coefficient of medium and p is the fluid pressure. For incompressible constituents of porous medium, $\mathfrak{b} = 1,\ s_\epsilon = 0$ [8]. Thus (1.1) reduces to

$$\zeta = \epsilon(\vec{u}). \tag{1.2}$$

The volumetric strain $\epsilon(\vec{u}) = \dfrac{\Delta V}{V}$ is known as Dilatation which is given by:

$$\epsilon(\vec{u}) = \nabla \cdot \vec{u},$$

and as such, the fluid content increment in (1.2) becomes

$$\zeta = \nabla \cdot \vec{u}. \tag{1.3}$$

Now, on the contrast to standard elasticity theory, incompressibility of each components of the deformable porous medium doesn't mean that both solid displacement and fluid velocity are divergence-free. This follows from the fact that the medium undergoes deformation (infinitesimal) under the applied external loads[2, 15]. But the volumetric strain of solid constituents lead to the variations of the fluid content of the porous medium. We use the convention $\nabla \cdot \vec{u}$ positive to mean extension implying "gain" of fluid in the medium [29].

Following the theoretical and numerical results in [17], we developed a one dimensional problem of deformable porous medium with incompressible constituents, for which we utilize the analytic solution to demonstrate that the fluid velocity and power density approach infinity if the stress boundary conditions is not smooth enough in time, and no viscoelasticity is present in the solid component. Then we utilize dimensional analysis to identify parameters that affect the discharge velocity blow-up, and hence providing the direction for sensitivity analysis on the system, which can help in the experimental design in Tissue engineering. The rest of the paper is presented in the following sequence. A brief description of poro-viscoelastic model for flow through deformable porous media under boundary traction is carried out in section 2. In section 3, the one-dimensional case of the model, for which an explicit solution is obtained and a summary on well-posedness of the solution is presented. Section 4 focuses on the dimensionless form of the one-dimensional problem. Results of the effect of two different cases of boundary tractions, namely step pulse (discontinuous in time) and trapezoidal pulse (no time discontinuity) in the presence or absence of viscoelasticity on the fluid dynamics are presents in section 5 and section 6 respectively. Finally, we end this paper by drawing general conclusions and by presenting some future development.

2. General Poro-viscoelastic Model Formulation and Description

Consider a deformable porous medium as a continuum and let $\Omega \subset \mathbb{R}^n,\ n = 1, 2, 3$ be a bounded open and connected domain with a Lipschitz continuous boundary $\partial\Omega$, and a unit outward normal $\hat{n}$ occupied by the medium. Let

$$\vec{u} : \Omega \times [0, T] \longrightarrow \mathbb{R}^n$$

be the displacement of the porous medium, and $\mathbf{E}(\vec{u})$ be the infinitesimal strain tensor for each time $t \in [0,T]$ with $T > 0$. Assuming that there is infinitesimal (small) deformation, full saturation of the mixture, incompressible mixture constituents, and quasi-static loading (negligible inertia), then the motion and mechanical response of the deformable porous medium under quasi-static loading are described by two set of equations, namely the balanced and constitutive equations.

2.1. Balance equations

These involve the balance of mass equation for the fluid content in the medium and the balance of linear momentum for the fluid-solid mixture of the medium. Here we consider the set of balance equations often ecountered in modeling fluid flow through porous media. Let $V_s(\vec{x},t)$ and $V_f(\vec{x},t)$ be the volumes occupied by the solid and fluid constituents at $(\vec{x},t) \in \Omega \times (0,T)$ respectively. In every volume $V(\vec{x},t)$ of a porous medium centred at $\vec{x} \in \Omega$ at time t, the volumetric fraction, $\phi(\vec{x},t)$ of the fluid constituents given by

$$\phi(x,t) = \frac{V_f(\vec{x},t)}{V(\vec{x},t)}. \tag{2.1}$$

is called the Porosity of the medium. If $V_p(\vec{x},t)$ is the pore volume of the medium, under full saturation condition, we have

$$V_p(\vec{x},t) = V_f(\vec{x},t),$$

and the volumetric fraction of the solid component of the medium is given by:

$$1 - \phi(\vec{x},t). \tag{2.2}$$

Following the assumptions, the motion of the poro-viscoelastic material is governed by the following balance of mass equation for the fluid component:

$$\frac{\partial \zeta}{\partial t} + \nabla \cdot \vec{v} = Q(x,t) \text{ in } \Omega \times (0,T) \tag{2.3}$$

and the balance of linear momentum equation for the fluid-solid mixture under Quasi-static loading:

$$\nabla \cdot \mathbf{T} + \vec{B} = 0 \text{ in } \Omega \times (0,T). \tag{2.4}$$

where, $\mathbf{T}$ is the stress tensor of the mixture (also called total stress), $\vec{v}$ is the discharge velocity, $\vec{B}$ is the body force per unit volume (source term of linear momentum), Q is the net volumetric fluid production rate (source term for fluid mass).

2.2. Constitutive equations

These include equations that quantify the total stress, discharge velocity and fluid contents respectively. For the purpose of our analysis, we consider the following constitutive equations

$$\begin{cases} \mathbf{T} = \mathbf{T}_e + \delta\mathbf{T}_v - p\mathbf{I} \\ \mathbf{T}_e = \lambda_e(\nabla \cdot \vec{u})\mathbf{I} + 2\mu_e\mathbf{E}(\vec{u}) \\ \mathbf{T}_v = \lambda_v\left(\nabla \cdot \dfrac{\partial \vec{u}}{\partial t}\right)\mathbf{I} + 2\mu_v\mathbf{E}\left(\dfrac{\partial \vec{u}}{\partial t}\right) \\ \zeta = \nabla \cdot \vec{u} \\ \vec{v} = -\mathbf{K}\nabla p, \end{cases} \tag{2.5}$$

where p is the Darcy fluid pressure, $\mathbf{I}$ is the identity tensor, $\mathbf{E}(\vec{u})$ is the infinitesimal strain tensor which is given by

$$\mathbf{E}(\vec{u}) = \frac{\nabla\vec{u} + (\nabla\vec{u})^T}{2}$$

which is the same as the symmetric part of the gradient of the displacement vector field $\vec{u}$, λ_e, and μ_e are the Lame elastic parameters, λ_v and μ_v are assumed to be viscoelastic parameters, $\mathbf{T}_e$, and $\mathbf{T}_v$ are the elastic and viscoelastic stress contributions respectively, the parameter $\delta \geq 0$ indicates the extent to which the model includes viscoelastic effect for the solid component, with $\delta = 0$, corresponds to the purely elastic case, and $\vec{v}$ is the fluid discharge velocity vector. The system of equations (2.3) to (2.5), with $\delta > 0$ characterise the motion of a *poro-viscoelastic medium* under Quasi-static loading in $\mathbb{R}^n$, $n = 1, 2, 3$. Further, the parameter $\mathbf{K}$ is the porous medium permeability tensor which may depend on space and dilation, that's

$$\mathbf{K} = \mathbf{K}(x, \nabla \cdot \vec{u})$$

[4, 11]. In addition to the system of partial differential equations (2.3) to (2.5), the following Dirichlet-Neumann boundary and initial conditions are considered:

2.2.1 Boundary conditions

Suppose that the boundary $\partial\Omega$ of Ω can be decomposed as $\partial\Omega = \Gamma_N \cup \Gamma_D$, with $\Gamma_D = \Gamma_{D,p} \cup \Gamma_{D,v}$, and possibly $\Gamma_D \cap \Gamma_v = \emptyset$ then

$$\begin{cases} \mathbf{T}\vec{n} = \vec{t_n}, \ \vec{v} \cdot \vec{n} = 0 \text{ on } \Gamma_N \times (0, T) \\ \vec{u} = \vec{0}, \ p = 0 \text{ on } \Gamma_{D,p} \times (0, T) \\ \vec{u} = \vec{0}, \ \vec{v} \cdot \vec{n} = \psi \text{ on } \Gamma_{D,v} \times (0, T) \end{cases} \tag{2.6}$$

Here the subscripts N and D indicates boundary conditions imposed on stress and displacement, whereas the subscripts p and v indicate conditions imposed on Darcy pressure and velocity, $\vec{t_n}$ is the force per unit area (surface traction), and ψ is a prescribed function of space and time.

2.2.2 Initial conditions

For the initial conditions, it is important to differentiate between two cases, namely viscoelastic case ($\delta > 0$) and purely elastic case ($\delta = 0$).
When $\delta > 0$, the balance of mass and linear momentum equations contain time derivatives, thus requiring an initial conditions on the whole displacement field $\vec{u}$. That's

$$\vec{u} = u_0 \text{ at } t = 0 \text{ in } \Omega \tag{2.7}$$

and when $\delta = 0$, only the increment in fluid content in the mass balance equation undergoes time derivative, and as such we have

$$\nabla \cdot \vec{u} = \epsilon_0 \text{ at } t = 0 \text{ in } \Omega, \tag{2.8}$$

where u_0, and ϵ_0 are prescribed initial data. Therefore equations (2.3) to (2.8) is known an Initial Boundary Value problem(IBVP) and the requirements (like elliptic regularity) for existence of solution for the case when $\delta = 0$ has been investigated in [17]. We found that most of the available theoretical studies based their investigation on poroelastic case with no account for structural viscoelasticity. More so, for constant medium permeability, the resulting coupling between elastic equations and fluid subproblem is linear and the existence, uniqueness and regularity of solutions have been studied by many authors [20, 25].

In the following section, we present a one dimensional poro-viscoelastic model, for which we can obtain an explicit solution and study the effect of structural viscoelasticity on the poromechanical response of the deformable porous medium to sudden changes in time-dependent mechanical load.

3. One-dimensional poro-viscoelastic problem formulation

Let the domain Ω be given as the open and bounded interval $(0, L) \subset \mathbb{R}$, with $0 < L < \infty$ and assuming in addition that there is no chemical reactions and no body force per unit volume, then $Q = \vec{B} = 0$ [6, 28]. Let $\sigma,\ \sigma_e,\ \sigma_v,\ u,\ v$ and K be the one-dimensional version of $\mathbf{T},\ \mathbf{T}_e,\ \mathbf{T}_v,\ \vec{u},\ \vec{v}$ and $\mathbf{K}$ respectively. In this case, the balance of mass and linear momentum equations in (2.3) to (2.4) become

$$\frac{\partial^2 u}{\partial t \partial x} + \frac{\partial v}{\partial x} = 0 \text{ in } (0, L) \times (0, T), \tag{3.1}$$

$$\frac{\partial \sigma}{\partial x} = 0 \text{ in } (0, L) \times (0, T) \tag{3.2}$$

and the constitutive equations in (2.5) yields

$$\sigma = \sigma_e + \delta \sigma_v - p \tag{3.3}$$

where,

$$\sigma_e = \lambda_e \left(\frac{\partial u}{\partial x}\right) + 2\mu_e \frac{\partial u}{\partial x}$$
$$= \alpha \frac{\partial u}{\partial x}, \text{ (where } \alpha = \lambda_e + 2\mu_e) \tag{3.4}$$
$$\delta\sigma_v = \delta\left(\lambda_v \left(\frac{\partial^2 u}{\partial x \partial t}\right) + 2\mu_v \left(\frac{\partial^2 u}{\partial x \partial t}\right)\right)$$
$$= \beta\left(\frac{\partial^2 u}{\partial x \partial t}\right), \text{ (where } \beta = \delta(\lambda_v + 2\mu_v)) \tag{3.5}$$

and

$$\zeta = \frac{\partial u}{\partial x}$$
$$v = -K\frac{\partial p}{\partial x}, \tag{3.6}$$

in $(0, L) \times (0, T)$ respectively with $0 \le \delta \le 1$.
Substituting (3.3), (3.4), and (3.5) into (3.2) we get

$$\alpha \frac{\partial^2 u}{\partial x^2} + \beta \frac{\partial^3 u}{\partial x^2 \partial t} - \frac{\partial p}{\partial x} = 0 \text{ in } (0, L) \times (0, T). \tag{3.7}$$

Now suppose that the deformable porous media with incompressible constituents under boundary traction is as shown in Figure (1).

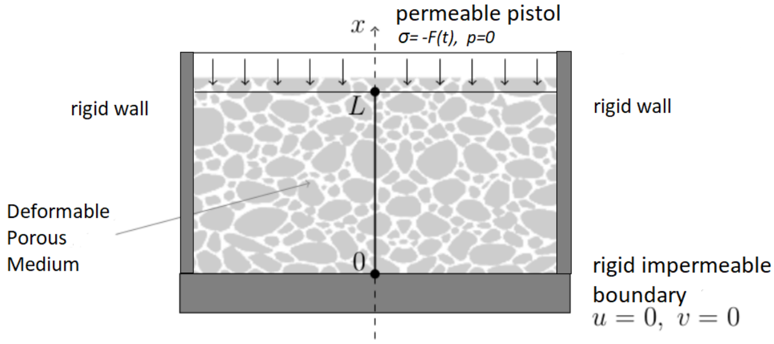

Figure 1: Representation of the one-dimensional poro-viscoelastic problem. The forcing term F(t) may have discontinuities in time.

Then we apply to the system of equations in (3.1) to (3.6), the following set of boundary and initial conditions which follows from (2.6)

$$u(0, t) = v(0, t) = 0 \; \forall \; 0 < t < T \tag{3.8}$$
$$p(L, t) = 0; \; \sigma(L, t) = -F(t) \; \forall \; 0 < t < T \tag{3.9}$$
$$u(x, 0) = 0 \; \forall \; 0 < x < L \tag{3.10}$$

where $F(t)$ is the boundary forcing term(traction force).
If $\beta = 0$ (absent of viscoelasticity) with the medium permeability K, and the boundary forcing term $F(t)$ are given constants, then the system (3.1) to (3.10) degenerates to the purely poro-elastic model studied experimentally and analytically in [26]. Besides, the boundary and initial conditions (3.8) to (3.10) considered correspond to the creep experimental test performed in confined compression test. Further, we rewrite the system of equations (3.1) to (3.10) in terms of the derivatives of the solid displacement u as follows:
First, by integrating both sides of (3.1), with respect to x we obtain

$$\frac{\partial u(x,t)}{\partial t} + v(x,t) = \eta(t), \tag{3.11}$$

where $\eta(t)$ is an arbitrary function of t. It follows from (3.8) that $\eta(t) = 0$, and thus (3.4) becomes

$$\begin{aligned} &\frac{\partial u(x,t)}{\partial t} + v(x,t) = 0 \\ \Longrightarrow\ & v(x,t) = -\frac{\partial u(x,t)}{\partial t} \end{aligned} \tag{3.12}$$

Substituting (3.12) into (3.6) yields

$$\frac{\partial u}{\partial t} = K\frac{\partial p}{\partial x}, \tag{3.13}$$

so that (3.7) becomes

$$\frac{\partial u}{\partial t} - K\alpha\frac{\partial^2 u}{\partial x^2} - K\beta\frac{\partial^3 u}{\partial t\partial x^2} = 0. \tag{3.14}$$

Now using (3.2) and (3.3), it follows from (3.9) that

$$\alpha\frac{\partial u(L,t)}{\partial x} + \beta\frac{\partial^2 u(L,t)}{\partial x\partial t} = -F(t). \tag{3.15}$$

Therefore, the system (3.1) to (3.10) reduces to the following non-linear initial boundary value problem which characterises a poro-viscoelastic medium in one-dimension:

$$\frac{\partial u}{\partial t} - K\alpha\frac{\partial^2 u}{\partial x^2} - K\beta\frac{\partial^3 u}{\partial t\partial x^2} = 0 \quad \text{in } (0,L)\times(0,T) \tag{3.16}$$

$$\alpha\frac{\partial u(L,t)}{\partial x} + \beta\frac{\partial^2 u(L,t)}{\partial x\partial t} = -F(t) \qquad 0 < t < T \tag{3.17}$$

$$u(0,t) = v(0,t) = 0 \qquad 0 < t < T \tag{3.18}$$

$$u(x,0) = 0 \qquad 0 < x < L \tag{3.19}$$

In addition to the system (3.2) − (3.10), we consider an important quantity associated with fluid-solid mixture called the fluid power density $\mathcal{P}(t)$ which is the time rate of energy transfer per unit volume of fluid. In our case, $\mathcal{P}(t)$ is given by:

$$\mathcal{P}(t) = \frac{1}{K}\int_0^L v(x,t)^2 dx, \tag{3.20}$$

where K is the medium permeability and $v(x,t)$ is the discharge velocity.

3.1. Formal solution

First, assume that medium permeability is a non-zero positive constant K_0, that's

$$K := K\left(x, \frac{\partial u}{\partial x}\right) \equiv K_0 > 0. \tag{3.21}$$

So that the system (3.19) to (3.19) becomes a Linear Initial Non-homogeneous Boundary Value Problem. We obtain a formal solution $u(x,t)$ to the resulting linear system using Fourier series expansion techniques as follows:
First, we homogenize the boundary conditions by setting

$$\varphi(x,t) = u(x,t) + G(x,t), \tag{3.22}$$

and suppose that for an auxilliary function $\mu(t)$

$$G(x,t) = \frac{\mu(t)}{\alpha}x, \tag{3.23}$$

which at $x = L$ satisfies

$$\alpha\frac{\partial G(L,t)}{\partial x} + \beta\frac{\partial^2 G(L,t)}{\partial x \partial t} = F(t). \tag{3.24}$$

Using (3.23), then (3.24) yields

$$\frac{d\mu(t)}{dt} + \frac{\alpha}{\beta}\mu(t) = \frac{\alpha}{\beta}F(t), \tag{3.25}$$

and by integrating factor techniques we get

$$\mu(t) = \frac{\alpha}{\beta}\int_0^t \exp\left(-\frac{\alpha}{\beta}(t-s)\right)F(s)ds = \frac{\alpha}{\beta}\exp\left(-\frac{\alpha}{\beta}t\right) \circledast F(t), \tag{3.26}$$

where $\circledast$ represents convolution. Hence, (3.22) becomes

$$\varphi(x,t) = u(x,t) + \frac{x}{\beta}\exp\left(-\frac{\alpha}{\beta}t\right) \circledast F(t), \tag{3.27}$$

which is the choice of function to homogenize the non-homogeneous problem. Therefore $\varphi(x,t)$ satisfies the problem

$$\frac{\partial \varphi}{\partial t} - K_0\alpha\frac{\partial^2\varphi}{\partial x^2} - K_0\beta\frac{\partial^3\varphi}{\partial t\partial x^2} = \frac{x}{\alpha}\mu'(t) \quad \text{in } (0,L)\times(0,T) \tag{3.28}$$

$$\alpha\frac{\partial\varphi(L,t)}{\partial x} + \beta\frac{\partial^2\varphi(L,t)}{\partial x\partial t} = 0 \qquad 0 < t < T \tag{3.29}$$

$$\varphi(0,t) = 0 \qquad 0 < t < T \tag{3.30}$$

$$\varphi(x,0) = 0 \qquad 0 < x < L, \tag{3.31}$$

where prime denotes the derivative of $\mu(t)$ with respect to t. Hence, the system (3.28) to (3.31) is a Linear Initial Homogeneous Boundary Value Problem and the associated eigenvalue problem is of the form:

$$y'' + \lambda y = 0,\ y(0) = y'(L) = 0. \tag{3.32}$$

$$\text{Find } y = y(x)\ 0 < x < L, \tag{3.33}$$

where the eigenvalues λ_n and the eigenvectors $y_n(x)$ are respectively given by

$$\lambda_n = \frac{(2n-1)^2\pi^2}{4L^2}, \text{ and } y_n(x) = \sin\left(\frac{2n-1}{2L}\right)\pi x,\ \forall n \in \mathbb{N}. \tag{3.34}$$

To the system of equations (3.28) to (3.31), we obtain a solution of the form

$$\varphi(x,t) = \sum_{n=1}^{\infty} a_n(t)y_n(x), \tag{3.35}$$

where $a_n(t)$ is the series' coefficients to be determined. For $f(x) = x,\ 0 < x < L$, the Fourier series expansion is

$$f(x) = \frac{2}{L}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{\lambda_n}y_n(x) \tag{3.36}$$

Substituting (3.35), and (3.36) into (3.28), we obtain

$$\sum_{n=1}^{\infty} a_n'(t)y_n(x) - K_0\sum_{n=1}^{\infty} a_n(t)y_n''(x) - K_0\beta\sum_{n=1}^{\infty} a_n'(t)y_n''(x) = \frac{2}{\alpha L}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{\lambda_n}y_n(x)\mu'(t) \tag{3.37}$$

$$\Longrightarrow \sum_{n=1}^{\infty} y_n(x)\big[(1+K_0\beta\lambda_n)a_n'(t) + \alpha K_0\lambda_n a_n(t)\big] = \sum_{n=1}^{\infty}\frac{2(-1)^{n-1}}{\alpha L\lambda_n}y_n(x)\mu'(t). \tag{3.38}$$

Using the uniqueness of Fourier series expansion [19], it follows from (3.38) that for any $n \in \mathbb{N}$

$$(1 + K_0\beta\lambda_n)a_n'(t) + \alpha K_0\lambda_n a_n(t) = \frac{2(-1)^{n-1}}{\alpha L\lambda_n}\mu'(t), \tag{3.39}$$

which is a first order ordinary differential equation in a_n, with initial condition $a_n(0) = 0$. By integrating factor techniques, we get

$$a_n(t) = \frac{2(-1)^{n-1}}{\alpha L\lambda_n(1 + K_0\beta\lambda_n)} \exp\left(-\frac{\alpha K_0\lambda_n}{1 + K_0\beta\lambda_n}t\right) \circledast \mu'(t), \tag{3.40}$$

so that (3.35) becomes

$$\varphi(x,t) = \sum_{n=1}^{\infty} \frac{2(-1)^{n-1}y_n(x)}{\alpha L\lambda_n(1 + K_0\beta\lambda_n)} \exp\left(-\frac{\alpha K_0\lambda_n}{1 + K_0\beta\lambda_n}t\right) \circledast \mu'(t). \tag{3.41}$$

Substituting (3.41) into (3.27), we get the solution

$$\begin{aligned} u(x,t) = -\frac{2}{\alpha L}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}y_n(x)}{\lambda_n}\Big(\mu(t) \\ - \frac{1}{(1 + K_0\beta\lambda_n)}\int_0^t \exp\big(-\frac{K_0\alpha\lambda_n}{1 + K_0\beta\lambda_n}(t-s)\big)\mu'(s)ds\Big) \end{aligned} \tag{3.42}$$

Taking (3.25) as identity coupled with integration by parts, we can rewrite (3.42) as follows:

$$u(x,t) = -\frac{2K_0}{L}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}y_n(x)}{(1 + K_0\beta\lambda_n)} \exp\left(-\frac{K_0\alpha\lambda_n}{1 + K_0\beta\lambda_n}t\right) \circledast F(t) \tag{3.43}$$

The expression in (3.43) is the formal solution to the problem in (3.16) – (3.19), with medium permeability given by (3.21), and $\beta > 0$.

3.2. Well-posedness of the Solution $u(x,t)$

Here we provide a summary that the solution $u(x,t)$ obtained for the case $\beta > 0$ actually solves the problem in (3.19) to (3.19) in a well-defined functional space. Let us introduce some notations. The symbol $H^k(\Omega)$ for $\Omega \subset \mathbb{R}^n$ open and for any integer $k \geq 0$ denotes the usual Sobolev space,

$$H^k(\Omega) = \{v \in \mathcal{L}^2(\Omega) : D^i v \in \mathcal{L}^2(\Omega)\ \forall |i| \leq k\}$$

equipped with the norm

$$\|v\|_k = \sqrt{\sum_{|i|\leq k} D^i v^2_{\mathcal{L}^2(\Omega)}}.$$

For real number $1 \leq p < \infty$, $\mathcal{L}^p(\Omega)$ denotes the space of strongly measurable continuous functions f on Ω such that the norm

$$\|f\|_{\mathcal{L}^p(\Omega)} = \left(\int_\Omega |f|^p\right)^{\frac{1}{p}} < \infty,$$

and for $p = \infty$ the norm is given by

$$\|f\|_{\mathcal{L}^\infty(\Omega)} = \max_{\vec{x}\in\Omega} f(\vec{x})$$

[1, 31]. For $\Omega = (0, L) \subset \mathbb{R}$, consider the real Sobolev space H given by

$$H := H^0(0, L) = \mathcal{L}^2(0, L),$$

which is a Hilbert space when equipped with an inner product

$$\langle f, g\rangle_H = \int_0^L fg dx \; \forall f, g \in H. \tag{3.44}$$

This inner product induces the norm

$$\|f\|_H = \left(\int_0^L f^2 dx\right)^{\frac{1}{2}} = \sqrt{\langle f, f\rangle_H}, \tag{3.45}$$

so that $(H, \|\cdot\|_H)$ becomes a Banach Space [1].

On the application of Gram-Schmidt Process to the linearly independent set $\{y_n(x)\}_{n=1}^{\infty}$ we obtain the orthonormal sequence $\mathcal{C} = \left\{\sqrt{\frac{2}{L}} y_n(x)\right\}_{n=1}^{\infty}$ in H [14]. Clearly, it follows from the linear independence property of orthonormal sequence that $\mathcal{C}$ span the inner product space H. The space of distributions on $(0, L)$ is denoted by $\mathcal{D}'(0, L)$. Using the basis $\mathcal{C}$ for H, it follows from the Convergence Theorem (3.5-2) in [14] that any $v \in \mathcal{D}'(0, L)$ belongs to H if and only if

$$v(x) = \sum_{n=1}^{\infty} c_n y_n(x) \tag{3.46}$$

(converging in $\mathcal{D}'(0, L)$,) with the scalars c_n satisfying

$$\sum_{n=0}^{\infty} |c_n|^2 < \infty.$$

Further, let

$$\mathcal{V} = \{v \in H : v' \text{ exist, } v' \in H \text{ and } v(0) = 0\},$$

and equip $\mathcal{V}$ with the inner product

$$\langle f, g\rangle_{\mathcal{V}} = \langle f', g'\rangle_H,\ \forall f, g \in \mathcal{V}. \tag{3.47}$$

Clearly, the space $(\mathcal{V}, \langle\cdot,\cdot\rangle_{\mathcal{V}})$ is a Hilbert space. By Poincaire Inequality [10], we obtain the norm $\|\cdot\|_{\mathcal{V}}$ as

$$\|v\|_{\mathcal{V}} = \|v'\|_H,\ \forall v \in \mathcal{V}. \tag{3.48}$$

As a consequence of Sobolev's inequality, $v \in \mathcal{V}$ implies that $v \in AC([0, L])$, the space of absolutely continuous function, so that the boundary condition $v(0) = 0$ is assumed in the strong sense. The sequence $\left\{\sqrt{\frac{2}{L\lambda_n}}y_n(x)\right\}_{n=1}^{\infty}$ is a Hilbert basis for $\mathcal{V}$, and any $v \in H$ is in $\mathcal{V}$ if the Fourier coefficients of its series expansion in (3.46) satisfies

$$\sum_{n=1}^{\infty} \lambda_n |c_n|^2 < \infty.$$

Therefore, for any $s \in \mathbb{R}$, we define a one-paremeter family of spaces, $\mathcal{V}^s$ as

$$\mathcal{V}^s = \left\{v \in \mathcal{D}'(0, L) : \exists\, a_k \in \mathbb{C},\ v(x) = \sum_{k=1}^{\infty} a_k y_n(x),\ \text{provided } \sum_{k=1}^{\infty} \lambda_k^s |c_k|^2 < \infty\right\}.$$

Next, we give the definition of a weak solution for the problem in (3.16) to (3.19) when $\beta > 0$.

Definition 3.1. A function

$$u : [0, L] \times [0, T] \longrightarrow \mathcal{V}$$
$$(x, t) \longmapsto u(x, t) := u$$

is a weak solution to problem (3.16) − (3.19) if

i) $u \in H^1([0, L] \times [0, T]; \mathcal{V})$, implies that $u, \dfrac{\partial u}{\partial t} \in \mathcal{L}^2([0, L] \times [0, T]; \mathcal{V})$

ii) For any $v \in \mathcal{V}$, and $t \in [0, T]$ pointwise a.e, we have

$$\left\langle\frac{\partial u}{\partial t}, v\right\rangle_H + K_0\left\langle\alpha u + \beta\frac{\partial u}{\partial t}, v\right\rangle_{\mathcal{V}} = -K_0 F(t) v(L) \tag{3.49}$$

iii) u(x,0)=0.

The initial condition (3.19) is satisfied by $u(x, t) \in \mathcal{V}$ in the pointwise sense according to condition (iii), since condition (i) implies that $u \in AC([0, L] \times [0, T]; \mathcal{V})$ [12]. The Boundary condition in (3.18) is included in the requirement that $u \in \mathcal{V}$, and the Boundary condition (3.17) is taking into account by condition (ii) for any test function $v \in \mathcal{V}$.

If $F(t) \in \mathcal{L}^2(0, T)$ then there exist a unique weak solution $u(x, t)$ according to Definition (3.1) to problem (3.16) to (3.19) [31, 5, 17, 28].

4. Dimensionless Form of Problem

In this section, we use dimensional analysis to provide the equivalence of problem (3.16) – (3.19) in dimensionless form so as to identify the combinations of geometrical and physical parameters that most influence the solution properties. Dimensional analysis is a mathematical technique that lean on the choice of a set of characteristic values that can be used to scale all variables in the problem. Here, we use the hat symbols to denote scaled (dimensionless) variables, and the square bracket to denote characteristic values of the variable. Then we obtain

$$\begin{cases} \hat{t} = \dfrac{t}{[t]} & \hat{x} = \dfrac{x}{[x]} & \hat{\beta} = \dfrac{\beta}{[\beta]} & \hat{\lambda}_n = \dfrac{\lambda_n}{[\lambda_n]} \\ \hat{\mathcal{P}} = \dfrac{\mathcal{P}}{[\mathcal{P}]} & \hat{F} = \dfrac{F}{[F]} & \hat{u} = \dfrac{u}{[u]} & \hat{v} = \dfrac{v}{[v]}. \end{cases} \tag{4.1}$$

It is crucial to note that there is no trivial selection for the characteristic values, and so this choice is not unique. In our case, the knowledge of the forcing term and the formal solution obtained will help us in the selection of these values.

4.1. Choice of characteristic values

Since the system is driving by the Boundary condition on the traction with the given function $F(t)$, we put

$$[F] = F_{\text{spec}}, \tag{4.2}$$

where F_{spec} is a given reference value. Following the expression in (3.34), we choose

$$[x] = L, \ [\lambda_n] = \frac{1}{L^2}. \tag{4.3}$$

From the explicit formula we have obtained for the solution we set

$$[\beta] = \frac{L^2}{K_0}, \ [t] = \frac{[K_0][\beta][\lambda_n]}{[K_0][\alpha][\lambda_n]} = \frac{[\beta]}{\alpha} = \frac{L^2}{K_0\alpha}, \ [u] = \frac{[K_0][F][t]}{[L]} = \frac{F_{\text{spec}}L}{\alpha} \tag{4.4}$$

According to (3.12), and (3.20) we select for

$$[v] = \frac{[u]}{[t]} = \frac{K_0 F_{\text{spec}}}{L}, \ [\mathcal{P}] = \frac{[L][v]^2}{[K_0]} = \frac{K_0 F_{\text{spec}}}{L}. \tag{4.5}$$

By chain rule and (4.1), we obtain

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial \hat{u}} \cdot \frac{\partial \hat{u}}{\partial \hat{t}} \cdot \frac{\partial \hat{t}}{\partial t} = \frac{[u]}{[t]} \frac{\partial \hat{u}}{\partial \hat{t}}, \tag{4.6}$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \hat{u}} \cdot \frac{\partial \hat{u}}{\partial \hat{x}} \cdot \frac{\partial \hat{x}}{\partial x} = \frac{[u]}{[x]} \frac{\partial \hat{u}}{\partial \hat{x}}, \tag{4.7}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial \hat{x}}\left(\frac{[u]}{[x]} \frac{\partial \hat{u}}{\partial \hat{x}}\right) \frac{\partial \hat{x}}{\partial x} = \frac{[u]}{[x]^2} \frac{\partial^2 \hat{u}}{\partial \hat{x}^2}, \tag{4.8}$$

$$\frac{\partial^2 u}{\partial t \partial x} = \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial \hat{t}}\left(\frac{[u]}{[x]} \frac{\partial \hat{u}}{\partial \hat{x}}\right) \frac{\partial \hat{t}}{\partial t} = \frac{[u]}{[x][t]} \frac{\partial^2 \hat{u}}{\partial \hat{t} \partial \hat{x}}, \tag{4.9}$$

and

$$\frac{\partial^3 u}{\partial x^2 \partial t} = \frac{\partial}{\partial t}\left(\frac{\partial^2 u}{\partial x^2}\right) = \frac{\partial}{\partial \hat{t}}\left(\frac{[u]}{[x]^2} \frac{\partial^2 \hat{u}}{\partial \hat{x}^2}\right) \frac{\partial \hat{t}}{\partial t} = \frac{[u]}{[x]^2[t]} \frac{\partial^3 \hat{x}}{\partial \hat{u}^2 \partial t} \tag{4.10}$$

Substituting (4.6)−(4.10) and the scalings in (4.2)−(4.5) into the problem (3.16)−(3.19) yield the following dimensionless problem

$$\frac{\partial \hat{u}}{\partial \hat{t}} - \frac{\partial^2 \hat{u}}{\partial \hat{x}^2} - \hat{\beta} \frac{\partial^3 \hat{u}}{\partial \hat{t} \partial \hat{x}^2} = 0 \qquad \text{in } (0,1) \times (0, \hat{T}) \tag{4.11}$$

$$\frac{\partial \hat{u}(1,\hat{t})}{\partial \hat{x}} + \hat{\beta} \frac{\partial^2 \hat{u}(1,\hat{t})}{\partial \hat{x} \partial \hat{t}} = -\hat{F} \qquad 0 < \hat{t} < \hat{T} \tag{4.12}$$

$$\hat{u}(0,\hat{t}) = 0 \qquad 0 < \hat{t} < \hat{T} \tag{4.13}$$

$$\hat{u}(\hat{x},0) = 0 \qquad 0 < \hat{x} < 1, \tag{4.14}$$

where $\hat{T} = T[t]$. Using the scaling system, we get the dimensionless form of the solid displacement $u(x,t)$ in (3.43) as

$$\hat{u}(\hat{x},\hat{t}) = -2 \sum_{n=1}^{\infty} \frac{(-1)^{n-1} y_n(\hat{x})}{1 + \hat{\beta}\hat{\lambda}_n} \exp\left(-\frac{\hat{\lambda}_n}{1 + \hat{\beta}\hat{\lambda}_n} \hat{t}\right) \circledast \hat{F}(\hat{t}), \tag{4.15}$$

the dimensionless form of discharge velocity in (3.12) as

$$\hat{v}(\hat{x},\hat{t}) = -\frac{\partial \hat{u}(\hat{x},\hat{t})}{\partial \hat{t}} = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n-1} y_n(\hat{x})}{1 + \hat{\beta}\hat{\lambda}_n} \left(\hat{F}(\hat{t}) - \frac{\hat{\lambda}_n}{1 + \hat{\beta}\hat{\lambda}_n} \exp\left(-\frac{\hat{\lambda}_n}{1 + \hat{\beta}\hat{\lambda}_n} \hat{t}\right) \circledast \hat{F}(\hat{t})\right), \tag{4.16}$$

and the dimensionless form of the fluid power density in (3.20) as

$$\hat{\mathcal{P}}(\hat{t}) = \int_0^1 \hat{v}(\hat{x},\hat{t})^2 d\hat{x} = 2 \sum_{n=1}^{\infty} \frac{1}{(1 + \hat{\beta}\hat{\lambda}_n)^2} \left(\hat{F}(\hat{t}) - \frac{\hat{\lambda}_n}{1 + \hat{\beta}\hat{\lambda}_n} \exp\left(-\frac{\hat{\lambda}_n}{1 + \hat{\beta}\hat{\lambda}_n} \hat{t}\right) \circledast \hat{F}(\hat{t})\right)^2 \tag{4.17}$$

The characteristic value chosen for velocity in (4.5) is the characteristic velocity induced by the external load of magnitude P_{spec} which is in contrast to what was utilized in [26]. The justification is that we are interested in accessing the influence of an external load on the poromechanical response of the medium.

5. Dimensionless step pulse load $\hat{F}(\hat{t})$

Here, we suppose that the forcing term (Boundary traction) $\hat{F}(\hat{t})$ be the dimensionless unit step defined by

$$\hat{F}(\hat{t}) = H(\hat{t}) = \begin{cases} 0, \text{ if } \hat{t} < 0 \\ 1, \text{ if } \hat{t} \geq 0 \end{cases}, \tag{5.1}$$

At the start of time $\hat{t} = 0$, there is discontinuity in the dimensionless load $\hat{F}(\hat{t})$ as shown in Figure (2).

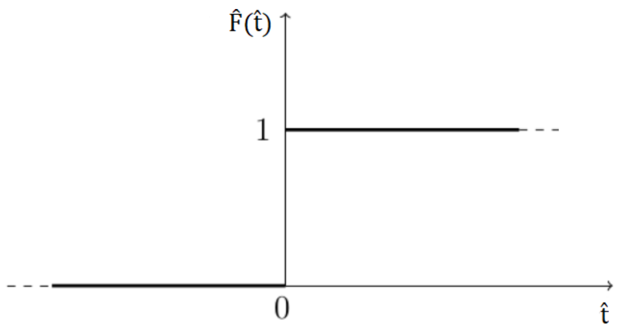

Figure 2: Diagrammatic representation of $\hat{F}(\hat{t})$ in (5.1)

Using (5.1), then (4.15), (4.16), and (4.17) respectively become

$$\hat{u}_{\hat{\beta}}(\hat{x}, \hat{t}) = -2 \sum_{n=1}^{\infty} \frac{(-1)^{n-1} y_n(\hat{x})}{\hat{\lambda}_n} \left(1 - \exp\left(-\frac{\hat{\lambda}_n}{1 + \hat{\beta}\hat{\lambda}_n}\hat{t}\right)\right), \tag{5.2}$$

$$\hat{v}_{\hat{\beta}}(\hat{x}, \hat{t}) = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n-1} y_n(\hat{x})}{1 + \hat{\beta}\hat{\lambda}_n} \exp\left(-\frac{\hat{\lambda}_n}{1 + \hat{\beta}\hat{\lambda}_n}\hat{t}\right), \tag{5.3}$$

and

$$\hat{\mathcal{P}}_{\hat{\beta}}(\hat{t}) = 2 \sum_{n=1}^{\infty} \frac{1}{(1 + \hat{\beta}\hat{\lambda}_n)^2} \exp\big(-\frac{\lambda_n}{1 + \hat{\beta}\hat{\lambda}_n}\hat{t}\big)^2. \tag{5.4}$$

Setting $\hat{\beta} = 0$, in (5.2) to (5.4), we get the dimensionless solutions $\hat{u}_0$, $\hat{v}_0$, and $\mathcal{P}_0$ for the dimensionless purely elastic problem. For $\hat{\beta} = 0$, and $\hat{\beta} = 1$, the space-time behaviour of $\hat{u}_{\hat{\beta}}$, $\hat{v}_{\hat{\beta}}$, and $\hat{\mathcal{P}}_{\hat{\beta}}$ is given in Figures (3), (4) and (5) respectively.

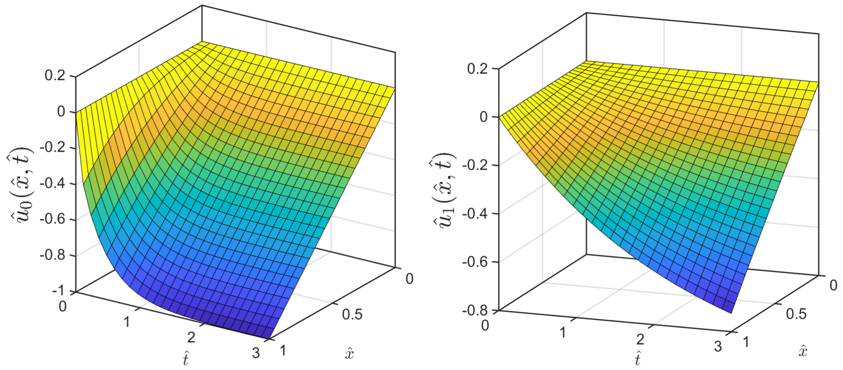

Figure 3: Dimensionless displacement as a function of $\hat{x}$, and $\hat{t}$. Left diagram: $\hat{\beta} = 0$. Right diagram: $\hat{\beta} = 1$

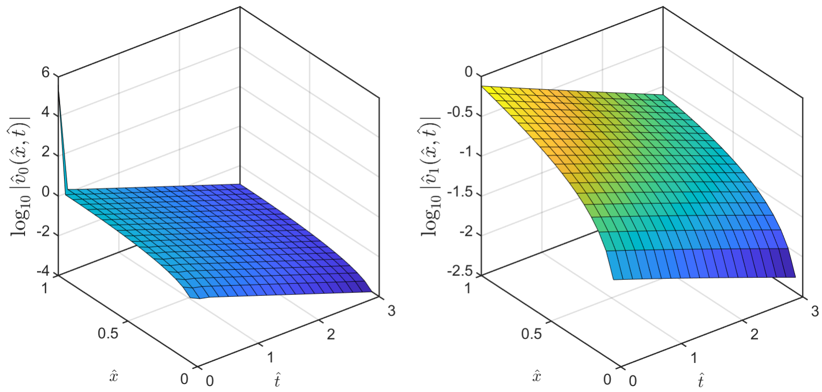

Figure 4: Dimensionless discharge velocity $\hat{v}_{\hat{\beta}}$ as a function of $\hat{x}$, and $\hat{t}$. Left plot: $\hat{\beta} = 0$. Right plot: $\hat{\beta} = 1$

The space time behaviour of $\hat{v}_{\hat{\beta}}$ and $\hat{\mathcal{P}}_{\hat{\beta}}$ respectively are provided on a logarithmic scale to see clearly the presence of a blow-up at $\hat{x} = 1$. However, it is not hard to see that by setting $\hat{\beta} = \hat{t} = 0$, we have

$$\hat{\mathcal{P}}_0(0) = 2\sum_{n=1}^{\infty} 1 = +\infty, \tag{5.5}$$

and

$$\hat{v}_0(\hat{x}, 0) = 2\sum_{n=1}^{\infty} (-1)^{n-1} y_n(\hat{x}). \tag{5.6}$$

In (5.6), for any $n \in \mathbb{N}$,

$$-1 \leq y_n(\hat{x}) \leq 1$$

and hence (5.6) lack pointwise convergence. From physical view points, the left plot in Figure (4) can be interpreted as; at the start of time, $\hat{t} = 0$, of the load, the discharge

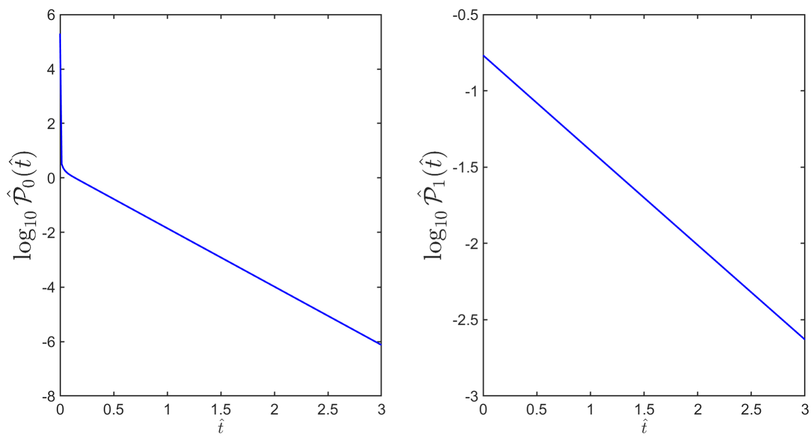

Figure 5: Dimensionless fluid power density $\hat{\mathcal{P}}_{\hat{\beta}}$ as a function of $\hat{t}$: Left diagram: $\hat{\beta} = 0$. Right diagram: $\hat{\beta} = 1$

velocity has high-frequency components whose superposition displays the following behaviours: It approaches infinity as $\hat{x}$ approaches 1, and hence leading to its blow-up, it relaxes towards zero as $\hat{x}$ approaches 0, it has an amorphous structure for $0 < \hat{x} < 1$. Suppose that viscoelasticity effect is present (see right plot of Figure (4)) then the dimensionless discharge velocity at the start of time $\hat{t} = 0$ becomes (putting $\hat{t} = 0$ in (5.3))

$$\hat{v}_{\hat{\beta}}(\hat{x},0) = 2\sum_{n=1}^{\infty}\frac{(-1)^{n-1}y_n(\hat{x})}{1+\hat{\beta}\hat{\lambda}_n} \tag{5.7}$$

$$= 8L^2\sum_{n=1}^{\infty}\frac{(-1)^{n-1}y_n(\hat{x})}{an^2-an+b}, \tag{5.8}$$

where $a = 4\hat{\beta}\pi^2$, and $b = 4L^2 + \hat{\beta}\pi^2$ and using Alternating series test [16], it is not difficult to see that the series in (5.8) is convergent. Also, it is continuous at $\hat{t} = 0$. We further illustrate these concepts in Figure (6) which gives the spatial distribution of $\hat{v}_{\hat{\beta}}(\hat{x},\hat{t})$ on logarithmic scale at $\hat{t} = 0$ for values of $\hat{\beta} \in [0,1]$. Therefore, at $\hat{x} = 1$ and $\hat{t} = 0$, we have the maximum value of $\hat{v}_{\hat{\beta}}$, in our case denoted by $\hat{v}_{max}(\hat{\beta})$ is attained, and it's given by

$$\hat{v}_{max}(\hat{\beta}) = \max_{0\le\hat{x}\le 1}|\hat{v}_{\hat{\beta}}(\hat{x},\hat{t})| = \hat{v}_{\hat{\beta}}(1,0) = 2\sum_{n=1}^{\infty}\frac{1}{1+\hat{\beta}\hat{\lambda}_n}, \tag{5.9}$$

which provides further investigation of the blow-up in $\hat{v}(\hat{x},\hat{t})$ and its dependence on the structural viscoelasticity. From Figure (5), the power density is decreasing with increase

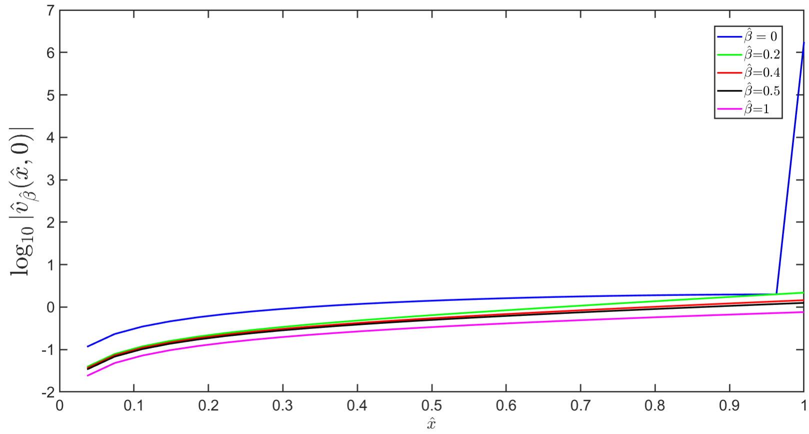

Figure 6: Dimensionless discharge velocity $\hat{v}_{\hat{\beta}}(\hat{x},0)$ as a function $\hat{x}$ to highlight the velocity blow-up at $\hat{x}=1$ when $\hat{\beta}=0$.

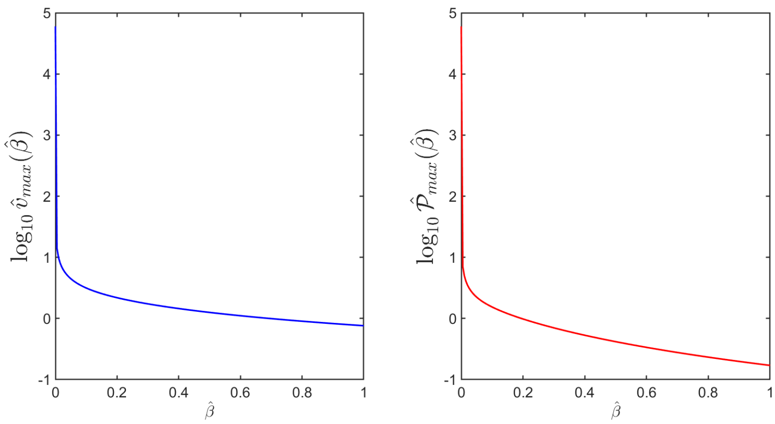

Figure 7: Left diagram: maximum discharge velocity as a function of $\hat{\beta}$. Right diagram: maximum power density as a function of $\hat{\beta}$.

in time and its maximum value, $\hat{\mathcal{P}}_{\max}(\hat{\beta})$ is attained at $\hat{t}=0$, that's

$$\hat{\mathcal{P}}_{max}(\hat{\beta}) = \max_{\hat{t}\leq 0}|\hat{\mathcal{P}}_{\hat{\beta}}(\hat{t})| = \hat{v}_{\hat{\beta}}(1,0) = 2\sum_{n=1}^{\infty}\frac{1}{(1+\hat{\beta}\hat{\lambda}_n)^2}. \tag{5.10}$$

This gives further investigation of the blow-up in $\hat{\mathcal{P}}(\hat{t})$ and its dependence on viscoelasticity. The behaviour of $\hat{v}_{\max}(\hat{\beta})$ and $\hat{\mathcal{P}}_{\max}(\hat{\beta})$ with respect to $\hat{\beta}$ is provided in Figure (7).

It is seen that the dimensionless parameter $\hat{\beta}$ is the main determinant of the blow-up in the dimensionless parameters $\hat{v}_{\max}(\hat{\beta})$ and $\hat{\mathcal{P}}_{\max}(\hat{\beta})$. On the other hand, it is important to note from the definition of $\hat{\beta}$ does depend uniquely on the viscoelastic coefficients (see equations (4.1), and (4.4)). That's small or large value of the medium permeability

K_0 and small or large value of the domain dimension L would have effect on the value of $\hat{\beta}$.

Using the analysis in (4.1), and (4.5) we get

$$\begin{cases} v_{\max}(\hat{\beta}) = \hat{v}_{\max}(\hat{\beta})[v] = \dfrac{F_{spec}K_0}{L}\hat{v}_{\max}(\hat{\beta}) \\ \\ \mathcal{P}_{\max}(\hat{\beta}) = \hat{\mathcal{P}}_{\max}(\hat{\beta})[\mathcal{P}] = \dfrac{F_{spec}^2 K_0}{L}\hat{\mathcal{P}}_{\max}(\hat{\beta}). \end{cases} \tag{5.11}$$

From (5.11), it can be seen that larger magnitude of the forcing term F_{spec} will give larger value of $v_{\max}$ and $\mathcal{P}_{\max}$ respectively. But the main factor that is in control of the blow-up still remains the dimensionless parameters $\hat{\beta} = \dfrac{K_0}{L^2}\beta$. We further illustrate this phenomenon in Figure (8), by setting $\dfrac{K_0}{L} = 1 Darcy/meter$, and $F_{spec} \in [10^{-4}, 10^4] newton/meter^2$.

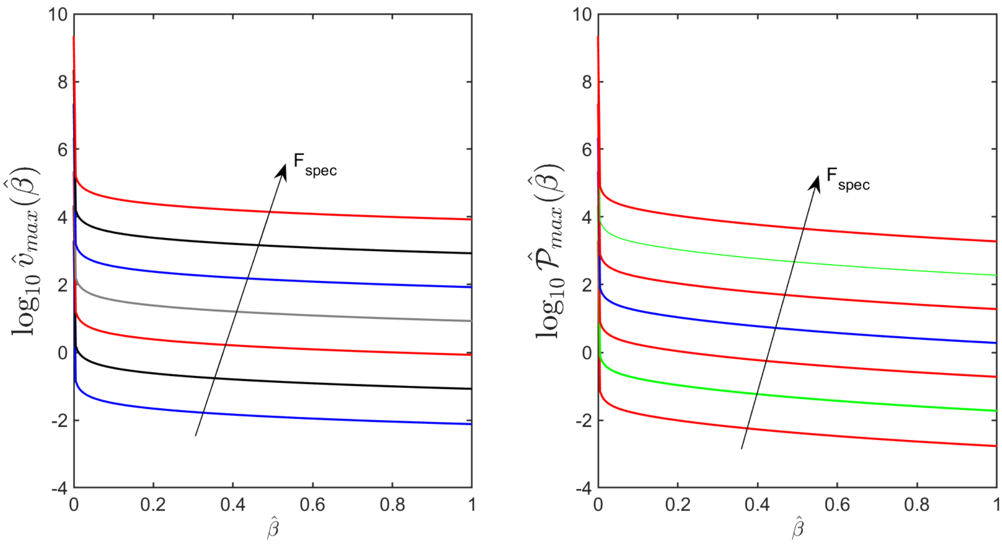

Figure 8: Left diagram: Dimensionless maximum discharge velocity as a function of $\hat{\beta}$. Right diagram: Dimensionless maximum power density as a function of $\hat{\beta}$. The arrows in both diagrams indicate increase in F_{spec}.

6. Dimensionless trapezoidal pulse load $\hat{F}(\hat{t})$

Here we consider a forcing term $\hat{F}(\hat{t})$ which is given by a dimensionless trapezoidal pulse of amplitude $a > 0$, in which switching on and off of the signal are characterized by linear ramp functions. Let $\hat{\xi}, \hat{\tau} \in \mathbb{R}$, be positive, then we take

$$\hat{F}(\hat{t}) = \begin{cases} 0, & \text{if } \hat{t} < 0 \\ \dfrac{a\hat{t}}{\hat{\xi}}, & \text{if } 0 \leq \hat{t} < \hat{\xi} \\ a, & \text{if } \hat{\xi} \leq \hat{t} < \hat{\tau} + \hat{\xi} \\ a\left(\dfrac{\hat{\tau} + 2\hat{\xi} - \hat{t}}{\hat{\xi}}\right), & \text{if } \hat{\tau} + \hat{\xi} \leq \hat{t} < \hat{\tau} + 2\hat{\xi} \\ 0, & \text{if } \hat{t} \geq \hat{\tau} + 2\hat{\xi} \end{cases} \tag{6.1}$$

where $\hat{\xi}$ denotes the pulse rise or fall in time and $\hat{\tau}$ denotes the duration of the pulse as shown in Figure (9).

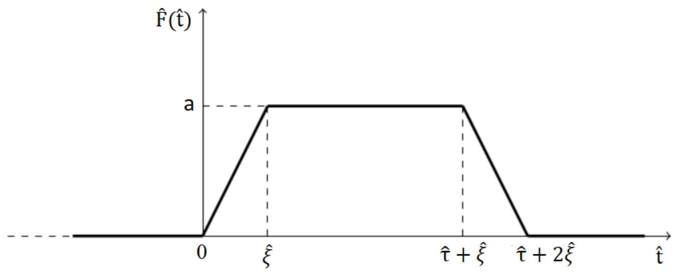

Figure 9: Diagrammatic representation of dimensionless $\hat{F}(\hat{t})$ defined in (6.1). In this case the signal switch on and off are given by linear ramp functions.

For conviniency, we let $\hat{v}^*_{\hat{\beta}}(\hat{x}, \hat{t})$ and $\hat{\mathcal{P}}^*_{\hat{\beta}}(\hat{t})$ be the dimensionless discharge velocity and dimensionless fluid power density respectively which result from the application of the dimensionless trapezoidal-pulse driving term $\hat{F}(\hat{t})$. On the application of Heaviside function $H(\hat{t})$, (6.1) can be expressed as follows:

$$\begin{aligned} \hat{F}(\hat{t}) = \frac{a}{\hat{\xi}}\big(&\hat{t}H(\hat{t}) - (\hat{t} - \hat{\xi})H(\hat{t} - \hat{\xi}) \\ &- (\hat{t} - \hat{\tau} - \hat{\xi})H(\hat{t} - \hat{\tau} - \hat{\xi}) - (\hat{t} - \hat{\tau} - 2\hat{\xi})H(\hat{t} - \hat{\tau} - 2\hat{\xi})\big). \end{aligned} \tag{6.2}$$

Now to compute $\hat{v}^*_{\hat{\beta}}(\hat{x}, \hat{t})$, first we make the following observations: Using the expression in (5.2), we obtain

$$\int_0^{\hat{t}} \hat{u}_{\hat{\beta}}(\hat{x}, s)ds = -2\sum_{n=1}^{\infty} \frac{(-1)^{n-1} y_n(\hat{x})}{1 + \hat{\beta}\hat{\lambda}_n}\left(\hat{t} - \frac{1 + \hat{\beta}\hat{\lambda}_n}{\hat{\lambda}_n} + \frac{1 + \hat{\beta}\hat{\lambda}_n}{\hat{\lambda}_n}\exp\left(-\frac{\hat{\lambda}_n}{1 + \hat{\beta}\hat{\lambda}_n}\hat{t}\right)\right), \tag{6.3}$$

and taking $\hat{F}(\hat{t}) = \hat{t}H(\hat{t})$, it follows from the expression in (4.15) that

$$\hat{u}(\hat{x}, \hat{t}) = \int_0^{\hat{t}} \hat{u}_{\hat{\beta}}(\hat{x}, s)ds, \tag{6.4}$$

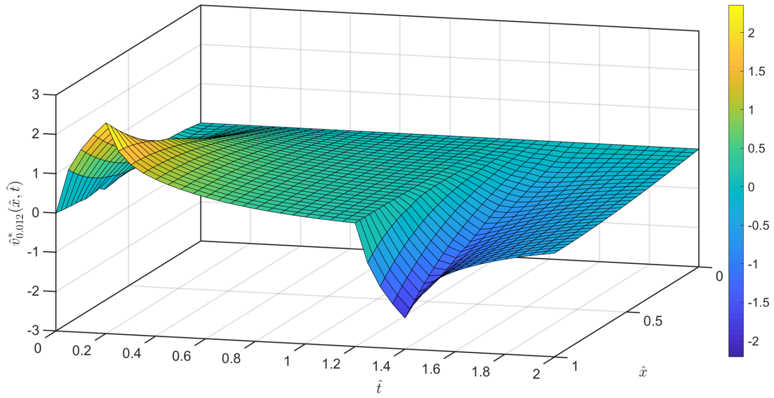

Figure 10: Dimensionless discharge velocity, $\hat{v}^*_{\hat{\beta}}(\hat{x},\hat{t})$ for $\hat{\beta} = 0.012$, $\hat{\xi} = 0.21$, $a = 1$ and $\hat{\tau} = 1$

so that using (3.12) the discharge velocity becomes $-\hat{u}_{\hat{\beta}}(\hat{x},\hat{t})$. Hence, applying the linear superposition principle, $\hat{v}^*_{\hat{\beta}}(\hat{x},\hat{t})$ is the linear combination of the discharge velocities corresponding to the several components of the pulse, that's

$$\begin{aligned}\hat{v}^*_{\hat{\beta}}(\hat{x},\hat{t}) =& \frac{a}{\hat{\xi}}\big(- \hat{u}_{\hat{\beta}}(\hat{x},\hat{t})H(\hat{t}) + \hat{u}_{\hat{\beta}}(\hat{x},\hat{t}-\hat{\xi})H(\hat{t}-\hat{\xi}) + \hat{u}_{\hat{\beta}}(\hat{x},\hat{t}-\hat{\tau}-\hat{\xi})H(\hat{t}-\hat{\tau}-\hat{\xi}) \\ &+ \hat{u}_{\hat{\beta}}(\hat{x},\hat{t}-\hat{\tau}-2\hat{\xi})H(\hat{t}-\hat{\tau}-2\hat{\xi})\big). \end{aligned} \tag{6.5}$$

Using (3.20) and (6.5) the dimensionless power density $\hat{\mathcal{P}}^*_{\hat{\beta}}(\hat{t})$ is computed as

$$\begin{aligned}\hat{\mathcal{P}}^*_{\hat{\beta}}(\hat{t}) =& \frac{a^2}{\hat{\xi}^2}\int_0^1 \big(\hat{u}_{\hat{\beta}}(\hat{x},\hat{t})H(\hat{t}) - \hat{u}_{\hat{\beta}}(\hat{x},\hat{t}-\hat{\xi})H(\hat{t}-\hat{\xi}) - \hat{u}_{\hat{\beta}}(\hat{x},\hat{t}-\hat{\tau}-\hat{\xi})H(\hat{t}-\hat{\tau}-\hat{\xi}) \\ &- \hat{u}_{\hat{\beta}}(\hat{x},\hat{t}-\hat{\tau}-2\hat{\xi})H(\hat{t}-\hat{\tau}-2\hat{\xi})\big)^2 d\hat{x}. \end{aligned} \tag{6.6}$$

For $\hat{\beta} \geq 0$, $a = 1$, and $\hat{\xi} > 0$, we report a typical space time behaviour of the dimensionless dischage velocity $\hat{v}^*_{\hat{\beta}}(\hat{x},\hat{t})$ and dimensionless fluid power density $\hat{\mathcal{P}}^*_{\hat{\beta}}(\hat{t})$ in Figures (10) and (11) respectively.

In this case, the maximum discharge velocity denoted by $\hat{v}^*_{\max}(\hat{\beta},\hat{\xi})$ is attained at $\hat{x} = 1$, and $\hat{t} = 0.21 = \hat{\xi}$, and it can be expressed as:

$$\hat{v}^*_{\max}(\hat{\beta},\hat{\xi}) = \max_{\substack{0\leq\hat{x}\leq 1\\ \hat{t}\geq 0}} \hat{v}^*_{\hat{\beta}}(\hat{x},\hat{t}) = \hat{v}^*_{\hat{\beta}}(1,\hat{\xi}) \tag{6.7}$$

$$= \frac{2}{\hat{\xi}}\sum_{n=1}^{\infty}\frac{1}{\hat{\lambda}_n}\left(1 - \exp\left(-\frac{\hat{\lambda}_n\hat{\xi}}{1+\hat{\lambda}_n\hat{\beta}}\right)\right), \tag{6.8}$$

and at $\hat{t} = \hat{\xi}$ the maximum dimensionless fluid power density, denoted by $\hat{\mathcal{P}}^*_{\max}(\hat{\beta},\hat{\xi})$ is

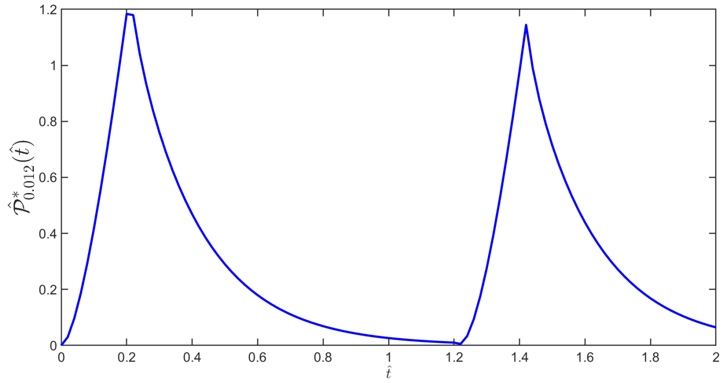

Figure 11: Dimensionless fluid power density, $\hat{\mathcal{P}}^*_{\hat{\beta}}(\hat{t})$ for $\hat{\beta} = 0.012,\ \hat{\xi} = 0.21,\ a = 1$ and $\hat{\tau} = 1$

attained and it can be expressed by

$$\hat{\mathcal{P}}^*_{\max}(\hat{\beta},\hat{\xi}) = \max_{t\geq 0} \hat{\mathcal{P}}^*_{\hat{\beta}}(\hat{t}) = \hat{\mathcal{P}}^*_{\hat{\beta}}(\hat{\xi}) \tag{6.9}$$

$$= \frac{4}{\hat{\xi}^2} \sum_{n=1}^{\infty} \Big(\frac{1 - \exp\big(- \frac{\hat{\lambda}_n \hat{\xi}}{1+\hat{\beta}\hat{\lambda}_n} \big)}{\hat{\lambda}_n} \Big)^2. \tag{6.10}$$

Figure (12) shows the behavior of $\hat{v}^*_{\max}$ with respect to $\hat{\beta}$ and $\hat{\xi}$. It is noted that for any $\hat{\beta} > 0$ and $\hat{\xi} > 0$,

$$\hat{v}^*_{\max}(\hat{\beta},\hat{\xi}) \leq \lim_{\hat{\xi}\to 0} \hat{v}^*_{\max}(\hat{\beta},\hat{\xi}), \tag{6.11}$$

and

$$\lim_{\hat{\xi}\to 0} \hat{v}^*_{\max}(\hat{\beta},\hat{\xi}) = \hat{v}_{\max}(\hat{\beta}),$$

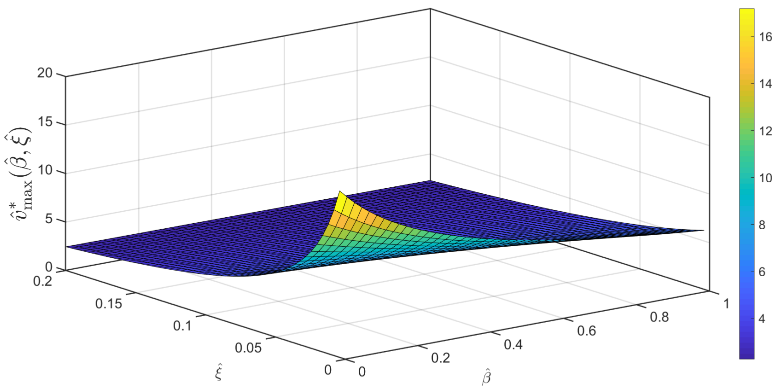

Figure 12: Dimensionless discharge velocity $\hat{v}^*_{\max}$ as a function of dimensionless pulse rise or fall in time $\hat{\xi}$ and dimensionless viscoelasticity $\hat{\beta}$

where $\hat{v}_{\max}(\hat{\beta})$ is given by (5.9). This is true since the trapezoidal pulse reduces to a rectangular pulse as $\hat{\xi} \to 0$. Hence, there is no blow-up in the viscoelastic case. Now for all $\hat{\beta} \geq 0$ and $\hat{\xi} > 0$,

$$\hat{v}^*_{\max}(\hat{\beta}, \hat{\xi}) \leq \hat{v}^*_{\max}(0, \hat{\xi}) = \frac{2}{\hat{\xi}} \sum_{n=1}^{\infty} \frac{1 - \exp(-\hat{\lambda}_n \hat{\xi})}{\hat{\lambda}_n}. \tag{6.12}$$

Therefore there is no blow-up also in the purely elastic case.

7. Conclusions and further developments

In this paper, we have utilized the theory of poro-viscoelasticity to model an incompressible deformable porous medium under quasi-static loading conditions in one dimension. The analytic solution obtained is used to identify a blow-up in the solution of the model of certain poro-viscoelastic problem often encountered in confined compression experiment. The analysis carried out in this paper gives the main factors that give rise to the blow-up, namely the absence of structural viscoelasticity and time-discontinuity of the boundary traction. Our findings showed that a very small viscoelasticity and a continuous time profile of the applied load will prevent blow-up even in the purely elastic case. The one-dimensional model considered in this paper can be used to generalize the mathematical analysis in [26]. More so, the findings of this study agree with the conclusions in [23] on the extreme sensitivity measure of the microstructural constitution of soft biological tissues to the active role of structural viscoelasticity. For future research, the mathematical analysis developed can be used to study confined compression experiment in tissue engineering, and a deformable porous medium with compressible constituents can be considered instead.

Acknowledgment

This project has received support from both Pan African University Institute for Basic Sciences Technology and Innovation (PAUSITI), and Japan International Cooperation (JICA).

Conflicts of interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] Adams, R. A. & Fournier, J. J. (2003). *Sobolev Spaces (Pure and applied mathematics; v. 140)*. Elsevier.

[2] Athanasiou, K. A. & Natoli, R. M. (2008). Introduction to continuum biomechanics. *Synthesis lectures on biomedical engineering*, *3*(1), 1–206.

[3] Banks, H., Bekele-Maxwell, K., Bociu, L., Noorman, M., & Guidoboni, G. (2017). Sensitivity analysis in poro-elastic and poro-visco-elastic models with respect to boundary data. *Quarterly of Applied Mathematics*, 75(4), 697–735.

[4] Berryman, J. G. (2003). Dynamic permeability in poroelasticity. *Stanford Exploration Project*, *113*, 1–454.

[5] Biot, M. (1941). General theory of three-dimensional consolidation. *Journal of applied physics*, 12(2), 155–164.

[6] Bowen, R. M. (1980). Incompressible porous media models by use of the theory of mixtures. *International Journal of Engineering Science*, *18*(9), 1129–1148.

[7] Cao, Y., Chen, S., & Meir, A. (2013). Analysis and numerical approximations of equations of nonlinear poroelasticity. *Discrete & Continuous Dynamical Systems-Series B*, *18*(5).

[8] Coussy, O. (2004). *Poromechanics*. John Wiley & Sons.

[9] Dominique, C., Gerbeau, J.-F., Sainte-Marie, J., & Clementel., I. E. V. (2010). A poroelastic model valid in large strains with applications to perfusion in cardiac modeling. *Computational Mechanics*, 46(1), 91–101.

[10] Evans, L. (2010). *Partial Differential Equations*. Graduate studies in mathematics. American Mathematical Society.

[11] Ghabezloo, S., Sulem, J., & Saint-Marc, J. (2009). Evaluation of a permeability–porosity relationship in a low-permeability creeping material using a single transient test. *International Journal of Rock Mechanics and Mining Sciences*, *46*(4), 761–768.

[12] Keener, J. P. (1995). *Principles of Applied Mathematics:*. Addison-Wesley Reading, MA;.

[13] Khanafer, K., AlAmiri, A., Pop, I., & Bull, J. L. (2008). *Flow and heat transfer in biological tissues: application of porous media theory*. Springer.

[14] Kreyszig, E. (1978). *Introductory functional analysis with applications*, volume 1. wiley New York.

[15] Lai, W. M., Rubin, D. H., Krempl, E., & Rubin, D. (2009). *Introduction to continuum mechanics*. Butterworth-Heinemann.

[16] Larson, R. & Edwards, B. H. (2009). *Calculus*. Cengage Learning.

[17] Lorena, B., Guidoboni, G., Sacco, R., & Webster, J. T. (2016). Analysis of nonlinear poro-elastic and poro-visco-elastic models. *Archive for Rational Mechanics and Analysis*, 222(3),1445–1519.

[18] Nia, H. T., Han, L., Li, Y., Ortiz, C., & Grodzinsky, A. (2011). Poroelasticity of cartilage at the nanoscale. *Biophysical journal*, 101(9), 2304–2313.

[19] O'Neil, P. V. (2011). *Beginning partial differential equations*, volume 85. John Wiley & Sons.

[20] Owczarek, S. (2010). A galerkin method for biot consolidation model. *Mathematics and Mechanics of Solids*, *15*(1), 42–56.

[21] Paola, C., Guidoboni, G., Harris, A., Prada, D., Sacco, R., & Terragni, S. (2014). A poroelastic model for the perfusion of the lamina cribrosa in the optic nerve head. *Mathematical biosciences*, 257, 33–41.

[22] Recha-Sancho, L., Moutos, F. T., AbellÃ , J., Guilak, F., & Semino, C. E. (2016). Dedifferentiated human articular chondrocytes redifferentiate to a cartilage-like tissue phenotype in a poly (epsilon-caprolactone) self-assembling peptide composite scaffold. *Materials*, 9(6), 472.

[23] Sack, I., Beierbach, B., Wuerfel, J., Klatt, D., Hamhaber, U., Papazoglou, S., Martus, P., & Braun, J. (2009). The impact of aging and gender on brain viscoelasticity. *Neuroimage*, *46*(3), 652–657.

[24] Selvadurai, A. P. S. (2003). On the mechanics of damage-susceptible poroelastic media. *In Key Engineering Materials. Trans Tech Publications.*, (Vol. 251, pp. 363–374).

[25] Showalter, R. E. (2000). Diffusion in poro-elastic media. *Journal of mathematical analysis and applications*, *251*(1), 310–340.

[26] Soltz, M. A. & Ateshian, G. A. (1998). Experimental verification and theoretical prediction of cartilage interstitial fluid pressurization at an impermeable contact interface in confined compression. *Journal of biomechanics*, *31*(10), 927–934.

[27] Terzaghi, K. (1925). Erdbaumechanik auf bodenphysikalischer grundlage. *F. Deuticke*, 101.

[28] Verri, M., Guidoboni, G., Bociu, L., & Sacco, R. (2017). The role of structural viscoelasticity in deformable porous media with incompressible constituents: applications in biomechanics. *American Institute for Mathematical Science*.

[29] Wang, H. F. (2000). Theory of linear poroelasticity.

[30] Yuan, Z. (2014). Model-based quantification of blood flow rate and oxygen consumption rate of biological tissues using image-guided near infrared light spectroscopy.

[31] Ženíšek, A. (1984). The existence and uniqueness theorem in biot's consolidation theory. *Aplikace matematiky*, *29*(3), 194–211.