Int. Journal of Math. Analysis, Vol. 4, 2010, no. 15, 727 - 736

On Numerical and Centre Values Range

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Abstract

This paper follows [1] in the quantitative study of Numerical Ranges introduced by Stampfli [9]. In particular, we consider a family of mutually orthogonal projections and investigate how a numerical range can be related to several other numerical ranges in a closed convex hull. We then introduce the centre valued range and show that if $\mathcal{U}$ is a W^*-algebra then for any A in $\mathcal{U}$ we can relate the norm of A and distance of A itself from its centre.

Mathematics Subject Classification: 47A12

Keywords: numerical range, centre valued range, representation, projection, convex hall

1 INTRODUCTION

Let $\mathcal{H}$ be a Hilbert space, T a bounded linear operator mapping $\mathcal{H}$ into $\mathcal{H}$ and $B(\mathcal{H})$ a set of bounded linear operators on $\mathcal{H}$. For any $T \in B(\mathcal{H})$ the numerical range $W(T)$ is the set

$$W(T) = \{\langle Tx, x\rangle : x \in \mathcal{H}, \|x\| = 1\}.$$

This set is convex, see [6], the classic Toeplitz-Hausdorff Theorem. The following properties of $W(T)$ are clear.

- $W(\alpha I + \beta T) = \alpha + \beta W(T), \quad \forall \alpha, \beta \in \mathcal{C}$
- $W(T^*) = \{\bar{\lambda} : \lambda \in W(T)\}$
- $W(U^*TU) = W(T)$ for any unitary operator U

Example 1.1 *Let $T \in B(\ell_2)$ be defined by $Tx = (x_2, x_3, ...,)$. Then $W(T)$ is an open disc of radius one, that is*

$$W(T) = \{z : |z| < 1\}$$

Lemma 1.2 *Let T be an operator on a two dimensional space. Then $W(T)$ is an ellipse whose foci are the eigenvalues of T.*

See Gustafson and Rao [10] for the proof.

It is clear that $B(\mathcal{H})$ is an algebra if multiplication for any two elements in $B(\mathcal{H})$ is pointwise defined. We shall also denote the dual of $B(\mathcal{H})$ by $B(\mathcal{H})^*$. For any element $T \in B(\mathcal{H})$ and identity $I \in B(\mathcal{H})$ the algebra numerical range $V(T)$ is given by

$$V(T) = \{f(T) : f(I) = 1 = \|f\|\}.$$

Definition 1.3 Let $\mathcal{A}$ be a C^*-algebra. The states of $\mathcal{A}$ are a class of linear functionals which maps positive values of an algebra to positive values of the same algebra.

$W(T)$ and $V(T)$ are identical, see [4]. The maximal numerical range of an operator $T \in B(\mathcal{H})$ is the set $W_0(T)$ where

$$W_0(T) = \{\lambda : \langle Tx_n, x_n \rangle \rightarrowtail \lambda, \|x_n\| = 1, \|Tx_n\| \rightarrowtail \|T\|\}.$$

This set was introduced by Stampfli [9]. If $\mathcal{U}$ is a C^*-algebra, for any $a \in \mathcal{U}$, the maximal numerical range is the set

$$\max V(a) = \{f(a) : f(I) = 1 = \|f\|, f(a^*a) = \|a\|^2\}.$$

It is clear that the maximal numerical range is convex. Stampfli, [9], also introduced the numerical range $W_\delta(T)$ given by

$$W_\delta(T) = clos\{\langle Tx, x \rangle : x \in \mathcal{H}, \|x\| = 1, \|Tx\| \geqslant \delta\}.$$

The set $W_\delta(T)$ is nonempty when $\|T\| > \delta$. It is also convex, see [2]. Since $B(\mathcal{H})$ is a unital C^*-algebra and $B(\mathcal{H})^*$ is its dual, we can also denote the set of states on $B(\mathcal{H})$ by $\xi(B(\mathcal{H}))$. For $T \in B(\mathcal{H})$ we can then define an algebra numerical range $V_\delta(T)$ by

$$V_\delta(T) = clos\{f(T) : f(I) = \|f\| = 1, f(T^*T) > \delta^2\}.$$

The sets $W_\delta(T)$ and $V_\delta(T)$ are identical, see [2]

Definition 1.4 An involution on an algebra $\mathcal{A}$ is a conjugate linear map $\mathcal{A} \to \mathcal{A}$ such that $(a^*)^* = a$ and $(ab)^* = b^*a^*$ for all $a, b \in \mathcal{A}$. An algebra with an involution is called a *-algebra.

A *-algebra $\mathcal{A}$ together with a complete submultiplicative norm such that $\|a^*\| = \|a\|, \forall a \in A$ is called a Banach *-algebra. A C^*-algebra is therefore a Banach *-algebra such that $\|a^*a\| = \|a\|^2 \ \forall a \in \mathcal{A}$. see Murphy, [8] and Bratteli and Robinson [3] for more details on C^*-algebra.

Let $\mathcal{A}$ and $\mathcal{B}$ be *-algebra. A mapping $\pi : \mathcal{A} \mapsto \mathcal{B}$ which satisfy the following condition is called a *-morphism

$$\pi(\alpha a + \beta b) = \alpha\pi(a) + \beta\pi(b) \qquad (1)$$

$$\pi(ab) = \pi(a)\pi(b), \quad \pi(a^*) = \pi(b)^*. \qquad (2)$$

The name morphism is usually reserved for mappings which only have properties (1) and (2).

Lemma 1.5 *Let $\mathcal{A}$ and $\mathcal{B}$ be a C^*-algebra and π a *-morphism of $\mathcal{A}$ into $\mathcal{B}$. It follows that*

- *π is positively preserving, that is $a \geqslant 0$ implies $\pi(a) \geq 0$,*
- *It is continuous and $\|\pi(a)\| \leq \|a\|, \ \forall a \in \mathcal{A}$.*

Proof. The proof of this can be found in Bratteli and Robinson [3] □

Definition 1.6 A *-morphism π from $\mathcal{A}$ to $\mathcal{B}$ is a *-isomorphism if it is one to one and onto, i.e. if the range of π is equal to $\mathcal{B}$ and if element of $\mathcal{B}$ is the image of another element of $\mathcal{A}$. Thus a *-morphism π of a C^*-algebra $\mathcal{A}$ onto a C^*-algebra $\mathcal{B}$ is a *-isomorphism if and only if $\ker\pi = \{0\}$, where $\ker\pi = \{a \in \mathcal{A} : \pi(a) = 0\}$

Definition 1.7 A representation of an algebra $\mathcal{A}$ is defined to be a pair $(\mathcal{H}, \pi)$, where $\mathcal{H}$ is complex Hilbert space and π is a *-isomorphism of $\mathcal{A}$ into $B(\mathcal{H})$. The representation $(\mathcal{H}, \pi)$ is said to be faithful if and only if, it is *-isomorphism between $\mathcal{A}$ and $\pi(\mathcal{A})$, that is if and and if $\ker\pi = \{0\}$.

Theorem 1.8 *Let $(\mathcal{H}, \pi)$ be a representation of a C^* -algebra $\mathcal{A}$. The representation is faithful if and only if it satisfies each of the following equivalent conditions*

- $\ker\pi = \{0\}$
- $\|\pi(a)\| = \|a\|, \forall a \in \mathcal{A}$
- $\pi(a) > 0, \forall a \in \mathcal{A}$.

See Bratteli and Robinson [3] for the proof.

Definition 1.9 If $(\mathcal{H}, \pi)$ is a representation of C^*-algebra $\mathcal{A}$ and $\mathcal{H}_1 \subset \mathcal{H}$, then $\mathcal{H}_1$ is said to be invariant or stable under π if $\pi(A)\mathcal{H}_1 \subseteq \mathcal{H}_1$, $\forall A \in \mathcal{A}$.

2 ALGEBRA NUMERICAL RANGE

Now consider a family of mutually orthogonal projections $(P_n) \subset \mathcal{U}$ with $\sum P_n = I$, I an identity in a C^*-algebra $\mathcal{U}$. Define numerical ranges

$$V = V\left(\mathcal{U}, \sum_{n=1}^{\infty} P_n a P_n\right) = \left\{f(\sum_{n=1}^{\infty} P_n a P_n) : f \in \mathcal{U}^*, f(I) = 1 = \|f\|\right\}$$

and

$$V_n = V(P_n \mathcal{U} P_n, P_n a P_n) = \{f(P_n a P_n) : f \in \varepsilon(P_n \mathcal{U} P_n)\}$$

respectively, where $\varepsilon(P_n \mathcal{U} P_n) = \{f \in (P_n \mathcal{U} P_n)^* : f(P_n) = 1 = \|f\|\}$. Then the following will be true.

Theorem 2.1

$$V = \overline{co}\left(\bigcup_{n=1}^{\infty} V_n\right).$$

Here $\overline{co}\left(\bigcup_{n=1}^{\infty} V_n\right)$ *is the closed convex hull of the sets* V_n*'s.*

Proof. Let $\lambda \in V$. Then there is a state $f \in \mathcal{U}$ such that $f(\sum_{n=1}^{\infty} P_n a P_n) = \lambda$. Take $\delta_k = f(\sum_{n=1}^{k} P_n a P_n)$ with $\lim_{k\to\infty} \delta_k = \lambda$. For each $n = 1, 2, ..., k$, define a functional g_n on $P_n \mathcal{U} P_n$ by restricting f to $P_n \mathcal{U} P_n$, that is

$$g_n(P_n a P_n) = f(P_n a P_n).$$

Clearly g_n is positive and linear. Let $0 \neq t_n = f(P_n)$. Clearly $0 < f(P_n) < 1$. Then $g_n = \frac{1}{t_n} \times f$ is a state on $P_n \mathcal{U} P_n$. So

$$\delta_k = \sum_{n=1}^{k} f(P_n a P_n) = \sum_{n=1}^{k} f(P_n) g_n(P_n a P_n)$$

and $\lim_{k\to\infty} \delta_k = \lambda$, implying that

$$\lambda = \sum_{n=1}^{\infty} f(P_n) g_n(P_n a P_n).$$

Since $g_n(P_naP_n) \in V_n$ for all n and considering the sequence

$$\frac{1}{\sum_{n=1}^{k} f(P_n)}\{f(P_1), f(P_2), \dots, f(P_k), 0, 0, 0, \dots\},$$

then

$$\begin{aligned}\lim_{k\to\infty} \frac{\sum_{n=1}^{k} f(P_n)g_n(P_naP_n}{\sum_{n=1}^{k} f(P_n)} &= \frac{\sum_{n=1}^{\infty} f(P_n)g_n(P_naP_n}{\sum_{n=1}^{\infty} f(P_n)} \\ &= \sum_{n=1}^{\infty} f(P_n)g_n(P_naP_n \in \bigcup_{n=1}^{\infty} V_n.\end{aligned}$$

To prove the converse, it is enough to show that $co(\bigcup_{n=1}^{\infty} V_n) \subseteq V$, since because V is closed, it follows that $\overline{co}(\bigcup_{n=1}^{\infty} V_n) \subseteq V$. So let $\lambda \in co(\bigcup_{n=1}^{\infty} V_n)$. Then $\lambda = \sum_{i=1}^{N} \alpha_i\lambda_i$, where $\sum_i \alpha_i = 1$, $\alpha_i \geq 0$ for each $0 \leq i \leq N$, $(\lambda_i)_{i=1}^{N} \subseteq \bigcup_{i=1}^{\infty} V_i$ and there is no loss of generality in assuming that $\lambda_i \in V_i$, $\forall i = 1, 2, ..., N$. Now define g on U by $g(x) = \sum_{i=1}^{N} \alpha_i f_i(P_ixP_i)$, then $g(I) = 1$. Therefore

$$\begin{aligned}g(\sum_{m=1}^{N} P_maP_m) &= \sum_{i=1}^{N} \alpha_i f_i(P_i(\sum_{m=1}^{N} P_maP_m)P_i) \\ &= \sum_{i}^{N} \alpha_i f_i(P_iaP_i) \\ &= \lambda.\end{aligned}$$

So $\lambda \in V$ as all $\lambda_i \in V_i$ and therefore $\overline{co}(\bigcup_{n=1}^{\infty} V_n) \subseteq V$. This completes the proof. □ Suppose P is a projection in a C^*-algebra $\mathcal{U}$, such that $P < I$, where I is the identity element. Consider $P\mathcal{U}P$ and define numerical range for any PaP in $P\mathcal{U}P$ by

$$V_\delta(P\mathcal{U}P, PaP) = clos\{f(PaP) : f \in \xi(P\mathcal{U}P),\ f[(PaP)^*(PaP)] \geq \delta^2\}$$

and

$$V_\delta(\mathcal{U}, PaP) = clos\{f(PaP) : f \in \xi(\mathcal{U}),\ f[(PaP)^*(PaP)] \geq \delta^2\}.$$

Then the following theorem will hold

Theorem 2.2 *If $P < I$, then*

$$co[\{0\} \bigcup V_\delta(P\mathcal{U}P, PaP)] = V_\delta(\mathcal{U}, PaP),$$

where $co[\{0\} \bigcup V_\delta(P\mathcal{U}P, PaP)]$ is the smallest convex set containing the sets $\{0\}$ and $V_\delta(P\mathcal{U}P, PaP)$.

Proof. Proving the inclusion

$$\{f(PaP) : f \in \xi(P\mathcal{U}P), f[(PaP)^*(PaP)] \geq \delta^2\} \subset V_\delta(\mathcal{U}, PaP).$$

is sufficient to imply that $V_\delta(P\mathcal{U}P, PaP)] \subset V_\delta(P\mathcal{U}P, PaP)]$. So, take $\lambda \in \{f(PaP) : f \in \xi(P\mathcal{U}P), f[(PaP)^*(PaP)] \geq \delta^2\}$. Then there exists a state $f \in \xi(P\mathcal{U}P)$ such that

$$f(PaP) = \lambda, f[(PaP)^*(PaP)] \geq \delta^2.$$

Define g on $\mathcal{U}$ by $g(x) = f(PxP)$. Clearly g is linear. Also

$$g(x^*x) = f(Px^*xP) \geq f[(PxP)^*(PaP)] \geq \delta^2.$$

So g is a positive linear functional and since $g(I) = f(P) = 1 = \|g\|$, we see that g is a state on $\mathcal{U}$. Since $g(a) = f(PaP) = \lambda$, we conclude that λ is in the set $V_\delta(\mathcal{U}, PaP)$.

Now let h_o be a state on $(I-P)\mathcal{U}(I-P)$. Then $h_o(I-P) = \|h\| = 1$. The identity in $(I-P)\mathcal{U}(I-P)$ is $(I-P)$. Define h on $\mathcal{U}$ by

$$h(x) = h_o[(I-P)x(I-P)].$$

This functional is positive and linear. Also $h(I) = h_o(I-P) = 1$ and so $\|h\| = 1$. Therefore h is a state on $\mathcal{U}$ and

$$h(PaP) = h_o[(I-P)(PaP)(I-P)] = 0.$$

So $0 \in V_\delta(\mathcal{U}, PaP)$. Since $V_\delta(\mathcal{U}, PaP)$ is convex, it follows that

$$co[\{0\} \bigcup V_\delta(P\mathcal{U}P, PaP)] \subseteq V_\delta(\mathcal{U}, PaP).$$

To prove the converse, it is enough to show that

$$\{f(PaP) : f\xi(P\mathcal{U}P), f[(PaP)^*(PaP)] \geq \delta^2\} \subseteq V_\delta(\mathcal{U}, PaP).$$

So let $\lambda \in V_\delta(\mathcal{U}, PaP)$. Then there exists $f \in \mathcal{U}^*$, $f(I) = 1 = \|f\|$ such that

$$f(PaP) = \lambda, f[(PaP)^*(PaP)] \geq \delta^2.$$

Define f_0 in $P\mathcal{U}P$ by retricting f to $P\mathcal{U}P$. Then the functional f_0 is positive and linear. If $\lambda = 0$, then we already have

$$0 \in co[\{0\} \bigcup V_\delta(P\mathcal{U}P, PaP)].$$

Suppose $\lambda \neq 0$. Let $t = f(P)$. Since $P < I$ and since by the Schwartz inequality

$$\lambda^2 = |f(PaP)|^2 \leq f(P)f(Pa(Pa)^*).$$

So $f(P) > 0$. That is, $0 < t < 1$. Then $\frac{1}{t} \times f_o$ is a state on $P\mathcal{U}P$. We also see that

$$\frac{1}{t}f[(PaP)^*(PaP)] = \frac{1}{t}f_o[(PaP)^*(PaP)] \geq \frac{1}{t}\delta^2 > \delta^2.$$

Hence

$$f(PaP) = \lambda = t(\frac{1}{t}f_o(PaP) + (1-t) \cdot 0.$$

That is

$$\lambda \in co[\{0\} \bigcup V_\delta(P\mathcal{U}P, PaP)].$$

So

$$co[\{0\} \bigcup V_\delta(P\mathcal{U}P, PaP)] \supseteq V_\delta(\mathcal{U}, PaP).$$

□

3 CENTRE VALUED RANGE

Let $\mathcal{U}$ be a $C^*-algebra$. We recall that the maximal numerical range $\max V(a)$, for $a \in \mathcal{U}$ is the set

$$\max V(a) = \{f(a) : f(I) = 1 = \|f\|, f(a^*a) = \|a\|\}.$$

If $\mathcal{U}$ is a $W^* - algebra$ with the centre $Z(\mathcal{U})$, Ω the maximal ideal space of $Z(\mathcal{U})$, ω any maximal ideal of $Z(\mathcal{U})$, $J(\omega)$ is the norm closure of

$$\sum_{i=1}^{N} Z_i X_i, \quad Z_i \in \omega,\ X_i \in \mathcal{U}.$$

The Glimm quotient is defined to be the set $\mathcal{U}(\omega) = \mathcal{U}/J(\omega)$. The canonical map of $\mathcal{U}$ into the Glimm quotient $U(\omega)$ is a homomorphism of $\mathcal{U}$ into the Glimm Quotient and for any $A \in \mathcal{U}, A(\omega)$ is the canonical image of A. The following result is due to J. Glimm [5].

Theorem 3.1 *If $A(\omega)$ is the canonical image of $A \in \mathcal{U}$, then*

$$A = \sup\{\|A(\omega)\| : \omega \in \Omega\}.$$

Proof. Assume thst f is a pure state in $\mathcal{U}(\omega)$. Then

$$\begin{aligned} \|A\| &= \|A^*A\| \\ &= \sup\{f(A^*A)\} \\ &= \sup\{f(A\omega)^*A(\omega))^{1/2} : \omega \in \Omega\} \\ &= \sup\{\|A^*(\omega)A(\omega)\|^{\frac{1}{2}} : \omega \in \Omega\} \\ &= \sup\{\|A(\omega) : \omega \in \Omega|\}. \end{aligned}$$

□ Larsen [7] established that $J(\omega)$ is a primitive ideal. It therefore follows that $\mathcal{U}(\omega)$ has a faithfull representation π_ω on some space $\mathcal{H}_\omega$.

Definition 3.2 Suppose ψ is a continuous linear map from $\mathcal{U}$ to its centre $Z(\mathcal{U})$. Let ψ also have the following properties

- $\psi(ZX) = Z\psi(X)\ \forall X \in \mathcal{U},\ Z \in Z(\mathcal{U})$,
- $\psi(X^*) = \psi(X)^*,\ \forall X \in \mathcal{U}$,
- $\|\psi\| = 1|\psi(I)|$.

We shall let E denote the set of all mappings satisfying the above conditions.

Definition 3.3 The centre valued range is the set

$$Z(\mathcal{U}) - V(A) = \{\psi(A) : \psi \in E\}.$$

The numerical range of an element $\pi_\omega(A(\omega))$ is therefore given by

$$V(\pi_\omega A(\omega))) = \{f(\pi_\omega(A(\omega))) : \|f\| = 1 = f(\pi_\omega(I(\omega)))\},$$

where $I(\omega)$ is an identity in $A(\omega)$.

The following theorem shows that the centre valued range $Z(\mathcal{U}) - V(A)$ of $A \in \mathcal{U}$ is equal to all $Z \in Z(\mathcal{U})$ such that Z belongs to the numerical range implemented by $\pi_\omega(A(\omega))$ in $B(\mathcal{H}_\omega)$.

Theorem 3.4 *Let $\mathcal{U}$ be a W^*-algebra with centre $Z(\mathcal{U})$, $A \in \mathcal{U}$, then*

$$Z(\mathcal{U}) - V(A) = \{Z \in Z(\mathcal{U}) : Z(\omega) \in V(\pi_\omega A(\omega)))\}.$$

For the proof of this theorem, see Glimm [5].

The maximal numerical range of $\pi_\omega(A(\omega)))$ in $B(\mathcal{H}_\omega)$ is given by

$$\max V(\pi_\omega(A(\omega))) = \{f(\pi_\omega(A(\omega))) : \|f\| = 1 = f(\pi_\omega(I(\omega))), f(\pi_\omega(A(\omega))^*\pi_\omega(A(\omega) = \|\pi_\omega(A(\omega\|^2\}.$$

The following theorem establishes the relationship between the maximal numerical centre valued range and the maximal numerical range.

Theorem 3.5 *Let $\mathcal{U}$ be a $W^* - algebra$ with centre $Z(\mathcal{U})$, $A \in \mathcal{U}$. Then*

$$\max Z(\mathcal{U}) - V(A) = \{Z \in Z(\mathcal{U}) : Z(\omega) \in \max V(\pi_\omega A(\omega))\}.$$

For the proof of this theorem see Glimm [5].

Theorem 3.6 *Let $\mathcal{U}$ be a W^*-algebra with centre $Z(\mathcal{U})$.*
If $0 \in \max Z(\mathcal{U}) - V(A)$, then for any $A \in \mathcal{U}$,

$$d(A, Z(\mathcal{U})) = \|A\|.$$

PROOF. Let $0 \in \max Z(\mathcal{U}) - V(A)$. Then there exists ψ which satisfies all those condtions in definition 3.2 such that $\psi(A) = 0$ and $\psi(A^*A)(\omega) = \|A(\omega)\|^2$ since $\psi(A) = 0, \psi(A^*) = 0$. Now for any $Z \in Z(\mathcal{U})$, we have

$$\begin{aligned}
\|A - Z\|^2 &= \|(A - Z)^*(A - Z)\| \\
&\geq \|\varphi(A^*A - Z^*A - A^*Z + Z^*Z)\| \\
&= \|\varphi(A^*A + Z^*Z\| \\
&= \sup\{|\varphi(A^*A(\omega) + \psi(Z^*Z(\omega)| : \omega \in \Omega\} \\
&\geq \sup \psi(A^*A)(\omega) : \omega \in \Omega \\
&= \sup\{\|A(\omega)\|^2 : \omega \in \Omega\} \\
&= \|A\|^2.
\end{aligned}$$

And so we note that

$$\inf\{\|A - Z\| : Z \in Z(\mathcal{U})\} \leq \|A\|.$$

Hence

$$d(A, Z(\mathcal{U})) = \|A\|.$$

The converse of this is not true. see, Agure [1]. □

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Received: June, 2009