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On Necessary Conditions for Scalars

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Abstract

In this paper, we give a characterization of scalar operators. In particular we show that a densely defined closed linear operator H acting on a reflexive Banach space X is scalar if it is of $(0, 1)$ type $\mathbb{R}$ and $\| f(H) \| \leq \| f \|_\infty$ for f in the algebra of smooth functions $\mathcal{U}$.

Keywords: Reflexive Banach space, A densely defined operator

1 Introduction

Suppose H is a closed densely defined operator on a Banach space X, whose spectrum is contained in $\mathbb{R}$ and there exist a $C > 0$ such that

$$\| (z - H)^{-1} \| \leq C \frac{< z >^\alpha}{| \mathcal{I} z |^{\alpha+1}} \tag{1}$$

for all $z \in i\mathbb{R}$ and some $(\alpha, \alpha+1) \geq 0$ then H is of $(\alpha, \alpha+1)$ type $\mathbb{R}$ [1].
Here, $< z >:= \sqrt{1+ \mid z \mid^2}$ and $\mathcal{I}z$ denotes the imaginary part of z.
A special case is a Hermitian operator on a Hilbert space. Scalar operators with real spectrum is called a pseudo-hermitian operator. In Hilbert space, abounded linear operator S is a pseudo-hermitian if and only if the group $\parallel e^{itS} \parallel \leq M < \infty$ for all $t \in \mathbb{R}$ [8].
If X is a reflexive Banach space then an operator $T \in B(X)$ is scalar spectral if it admits an integral representation with respect to countably additive projection valued measure or equivalently if it admits a $C(\sigma(T))$ functional calculus [6]. In particular, if T acts on a Hilbert space $\mathcal{H}$, then T admits $C(\mathbb{R})$ functional calculus if it is Hermitian. Generally, an operator acting on a reflexive Banach space is scalar if and only if it has a $C_o(\mathbb{R})$ functional calculus [5]. According to [10], If T is an operator with $\sigma(T) \subset \mathbb{R}$ and acting on a reflexive Banach space X, then T is scalar if and only if iH generates a uniformly bounded strongly continuous group. In [7], a functional calculus is given for a closed densely defined linear operators on a Banach space with $\sigma(H) \subseteq \mathbb{R}$ satisfying the resolvent estimate and for functions from weighted sobolev spaces. Here the calculus used is based on almost analytic extension to $\mathbb{C}$ of infinitely differentiable functions defined on $\mathbb{R}$ and the Helffer-Sjostrand formula [9]. Such calculus defines an algebra homomorphism. We now consider an intermediate topology $C_c^\infty(\mathbb{R}) \subset \mathcal{U} \subseteq C(\mathbb{R})$ such that $(\alpha, \alpha+1)$ type $\mathbb{R}$ operators admits $\mathcal{U}$ functional calculus. Here $C_c^\infty(\mathbb{R})$ is the space of smooth functions of compact support. For detailed information see [2].
For any $f \in \mathcal{U}$ the norm is defined as;

$$\parallel f \parallel_n := \sum_{r=0}^{n} \int \mid f^{(r)}(x) \mid < x >^{r-1} dx \tag{2}$$

where

$$\mid f^{(r)}(x) \mid := \mid \frac{d^r}{dx^r} f(x) \mid \leq c_r < x >^{\beta - r} \tag{3}$$

for all $x, \beta \in \mathbb{R}$ and $c_r > 0$.
It is shown in [2] that $\mathcal{U}$ is an algebra under pointwise multiplication.
The definition of $f(H)$ for $f \in \mathcal{U}$ originates from the version of Helffer and Sjöstrand [9] integral formula. Using this and the abstract result from [3], we show that a densely defined closed linear operator H acting on a reflexive Banach space X is scalar if it is of $(0,1)$ type $\mathbb{R}$ and $\parallel f(H) \parallel \leq \parallel f \parallel_\infty$

2 The $\mathcal{U}$ functional Calculus

The materials in this section has been taken from [3] and [4].
For any $f \in \mathcal{U}$ and $n \geq 0$ an almost analytic extension of f to $\mathbb{C}$ is defined;

$$\widetilde{f}(x+iy) := \sum_{r=0}^{n} \frac{f^{(r)}(x)(iy)^r}{r!} \tau\left(\frac{y}{< x >}\right) \tag{4}$$

where τ is a $\mathcal{C}_c^\infty(\mathbb{R})$ function such that $\tau(s) = 1$ if $\mid s \mid \leq 1$ and $\tau(s) \geq 2$.
It follows that for $f \in \mathcal{U}$, $\mid \frac{\partial}{\partial \overline{z}} \widetilde{f}(x,y) \mid = \mathcal{O}(\mid y \mid^n)$ as $\mid y \mid \rightarrow 0$ for a fixed x. Moreover we can find $c' \in \mathbb{R}$ such that

$$\mid \frac{\partial}{\partial \overline{z}} \widetilde{f}(x,y) \mid \leq c'(\mid y \mid^n) \tag{5}$$

as $z \rightarrow x \in \mathbb{R}$. If κ is a map such that $\kappa : \mathcal{U} \rightarrow B(X)$ then

$$f \rightarrow f(H) := -\frac{1}{\pi} \int_{\mathbb{C}} \frac{\partial \widetilde{f}}{\partial \overline{z}} (z-H)^{-1} dxdy \tag{6}$$

and it is proved in [3], that for $n > \alpha \geq 0$

- $f(H)$ is norm convergent with $\parallel f(H) \parallel \leq C_\alpha \parallel f \parallel_{n+1}$ for some $C_\alpha > 0$ and doesn't depend on τ;
- the mapping extends to a bounded algebra homomorphism;
- if $f \in \mathcal{U}$ and $f = 0$ on a neighbourhood of $\sigma(H)$ then $f(H) = 0$;
- if $z \in i\mathbb{R}$ then $\frac{1}{z-} \in \mathcal{U}$ and $f(\frac{1}{z-}) = (z-H)^{-1}$

For an operator H of $(\alpha, \alpha+1)$-type $\mathbb{R}$, we associate each element $f \in \mathcal{U}$ with an operator $f(H) \in B(X)$ given by (6)
In order to state our results, we need the following theorems and corollaries;

Theorem 2.1 *Let H be a bounded operator with $\sigma(H) \subseteq \mathbb{R}$, and $\parallel e^{iHt} \parallel \leq C(1+ \mid t \mid)^\alpha$ where α is a non negative integer. Then H is of $(\alpha, \alpha+1)$ type $\mathbb{R}$*
Proof. see[1]

corollary 2.2 *If $\alpha = 0$ then H is of $(0,1)$ type $\mathbb{R}$ and $\parallel e^{iHt} \parallel \leq C < \infty$. In particular H is a pseudo hermitian operator, and so it is a scalar operator.*

Theorem 2.3 *H is a generator of a C_o-contraction semi-group if and only if H is closed, densely defined and for each $\lambda > 0$, $\lambda \in \rho(H)$ and $\parallel (\lambda - H)^{-1} \parallel \leq \lambda^{-1}$*

Proof. see[1]

corollary 2.4 *If iH is a generator of a group of isometries $\{T(t)\}$ then for all $\lambda \in i\mathbb{R}$ with real $\lambda \neq 0$, $\lambda \in \rho(iH)$ and*

$$(\lambda - iH)^{-1} = \begin{cases} \int_0^\infty T(t)e^{-\lambda t}dt, & if\ R\lambda > 0; \\ -\int_0^\infty T(t)e^{-\lambda t}dt, & if\ R\lambda < 0; \end{cases}$$

Theorem 2.5 *If H is a Hermitian operator on a Hilbert space $\mathcal{H}$, then H is of $(0,1)$ type $\mathbb{R}$*

The proof of this theorem follows from the fact that since H is a Hermitian operator then obviously its spectrum is in $\mathbb{R}$ and so the resolvent set of H; $\rho(H) := \{z \in i\mathbb{R} : z - H : \mathcal{D}(H) \to X$ is bijective and $(z - H)^{-1} \in B(X)\}$. In particular $\| (z - H)^{-1} \| \leq C| \mathcal{I}z |^{-1}$ by (1); thus for all $z \in \rho(H)$, $R(z, H) := (z - H)^{-1}$ is a normal operator.

Theorem 2.6 *If H is of $(\alpha, \alpha + 1)$-type $\mathbb{R}$ for some $\alpha > 0$, then H admits $C_o^\infty(\mathbb{R})$ functional calculus.*

Proof. see [1]

Theorem 2.7 *If $f \in \mathcal{U}$ and H is Hermitian on a Hilbert space $\mathcal{H}$, then*

$$\| f(H) \| \leq \| f \|_\infty .$$

Proof. See[3]

3 Main Results

Theorem 2.8 *H is of $(0,1)$-type $\mathbb{R}$ with the constant $C = 1$ if and only if iH is a generator of a one parameter group of isometries on X.*

Proof.

Suppose H is of $(0,1)$-type $\mathbb{R}$ with $C = 1$, then from corollary (2.2), H is a Pseudo-Hermitian operator, and hence a scalar operator. It follows from (1) and (Theorem 2.1) that H is a generator of a one parameter group of isometries and so iH also generates a group of isometries. Since iH generates a group of isometries it follows that iH is densely defined. Also from corollary(2.4); we have that for $\lambda > 0$; $(\lambda - iH)^{-1}f$ is the Laplace transform of $T(t) = e^{itH}f$ given for f in the domain of iH. It follows from theorem (2.1) that $T(t)$ is a bounded operator. Conversely, suppose iH densely defined and theorem (2.1) holds, then $T(t)$ is a semigroup. From the uniform boundedness, $(T(t)_{t\geq 0})$ is uniformly bounded on compact intervals. From corollary(2.4) and density of $D(iH)$, imply that $(T(t))$ is strongly continuous. Hence iH is a generator of one parameter group T.

Theorem 2.9 *A densely defined linear operator H acting on a reflexive Banach space X, is scalar if it is of $(0,1)$-type $\mathbb{R}$ and $\| f(H) \| \leq \| f \|_\infty$ for each $f \in \mathcal{U}$*

Proof. Let H be an operator acting on Hilbert space $\mathcal{H}$ and $\sigma(H) \subseteq \mathbb{R}$, then H is A Pseudo Hermitian Operator. By theorem (2.5) it is of (0,1)-type $\mathbb{R}$ and by theorem (2.6) it is a scalar operator. Also iH generates a one parameter group by theorem 2.8, hence H admits a functional calculus given by (6). Since (6) is continuous by (1) and (5), the resolvent set is bounded. From theorem (2.7) we see that (6) converges absolutely. Since H is Hermitian, it follows by Riesz representation theorem that for $f \in \mathcal{U}$ there exist a complex Borel measure μ on $\sigma(H)$ such that

$$f(H) = \int_{\sigma(H)} f(z)\mu dz$$

and this completes the proof.

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References

[1] P O. Oleche, N. O Ongati and J. O Agure. Operators with slowly growing resolvents towards the spectrum. *International Journal of Pure and Applied Mathematics,*51(3):245-357, March 2009.

[2] P O. Oleche, N. O Ongati and J. O Agure. The algebra of smooth functions of rapid descent.*International Journal of Pure and Applied Mathematics,*52(2):163-176, 2009.

[3] P O. Oleche, J. O Agure. A functional Calculus for $(\alpha, \alpha + 1)$- type $\mathbb{R}$ Operators *Int. Journal of Math. Analysis,*vol.4,2010,no,737-760

[4] A.Batkai and E.Fasanga *The spectral mapping theorem for Davis' functional calculus AMS* (2000),1-7.

[5] H.R. Dowson. *Spectral Theory of Linear Operators. Academic Press, New York,* 1978.

[6] I. Doust and G.Lancien *The spectral type of sums of operators on non-Hilbertian Banach lattices.J.Aust. Math Soc.*84(2008),193-198.

[7] E.B. Davies, *The functional calculus. J. London Math. Soc.* 52(1995),166-176.

[8] S. Kantorovitz *On the characterization of spectral operators.Trans.Amer.Math.Soc.*111(1964),152-181.

[9] Helffer B. and Sjöstrand, J. *Euation de Schrodinger Operator avec magnetique et equation de Harper*,345 Of Lecture notes in Physics,SprinerVerlag,Berline, 1989.

[10] S.Kantorovitz. *Spectral theory of Banach space Operators,* volume 1012 of *Lecture notes in Mathematics*. Springer-Verlag, Berlin, 1989.

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