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Norms of Derivations Implemented by S-Universal Operators

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Abstract

Let H be a complex Hilbert space and $B(H)$ the algebra of all bounded linear operators on H. For $A, B \in B(H)$, we define inner derivations implemented by A, B respectively on $B(H)$ by $\Delta_A(X) = AX - XA$, $\Delta_B(X) = BX - XB$ and a generalized derivation by $\Delta_{A,B}(X) = AX - XB$, $\forall X \in B(H)$. We establish the relationship between the norms of Δ_A, Δ_B and $\Delta_{A,B}$ on $B(H)$, specifically, when the operators A, B are S - universal.

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1 Introduction

The study of elementary operators has continued to attract the attention of many researchers. Of special interest is the calculations of norms of these operators. See for instance [1, 2, 10]. Special classes of elementary operators include multiplication operators, inner derivations, generalized derivations, basic elementary operators among others. Surprisingly, the norm of a general elementary operator is still unresolved even for a 'simple' algebra such as $B(H)$, the algebra of bounded linear operators on a Hilbert space H. Several attempts that have been made have always restricted these operators to specific cases. In [1] the authors used spectral resolution theorem to calculate the norm of an elementary operator induced by normal operators in a finite dimensional Hilbert space. In [10], the norm of a basic elementary operator has been established.

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For symmetrised two-sided multiplication operators, Nyamwala recently in [2] constructed the operator $T_{PQ_k}E = PEQ_k + Q_kEP$ defined on a C*-algebra $C^*(P, Q_k, 1)$ generated by projections P and Q_k where E is an idempotent related to P and Q_k and determined its norm as well as Hilbert-Schmidt norm. These and much more examples provide enough evidence that the norms of these operators still calls for considerable attention.

On the other hand, the theory of inner derivations on norm ideals has been studied by several authors. See for instance [5, 7]. In this paper, we shall concentrate on a class of operators known as the S - universal operators. These operators were introduced by L. Fialkow in 1979, see [7]. Some reasonable exposition on these operators was done in 2001 by Barraa and Boumazgour, see [8, 9]. The structural properties of these operators are still not known though their 'beautiful' applications have recently been explored by the current authors and for details, see [5, 6].

In this paper, we purely concentrate on norms of inner and generalized derivations implemented by S - universal operators. Section 2 contains preliminaries. Here we include definitions and elementary concepts of derivations that will be vital in stating authoritatively the major results of the paper later.

The main results of this paper are in section 3. We establish the relationship between the norm of a generalized derivation implemented by $A, B \in B(H)$ and the norms of inner derivations implemented by $A \in B(H)$ and $B \in B(H)$ respectively. Finally, we give a necessary and sufficient condition that characterizes the above relationship. We then conclude our paper in section 4.

2 Preliminaries

In order to state our results in detail, we first recall some notations, definitions and results from literature. A derivation on an algebra $B(H)$ is a linear map $\Delta : B(H) \longrightarrow B(H)$ which satisfies $\Delta(AX) = \Delta(A)X + A\Delta(X)$ for all $A, X \in B(H)$. If we fix A, then a mapping defined by $\Delta_A(X) = AX - XA$ for all $X \in B(H)$ is called an inner derivation. Also for a fixed $A, B \in B(H)$, a mapping defined by $\Delta_{A,B}(X) = AX - XB$ for all $X \in B(H)$ is called a generalized derivation. We shall denote a derivation on $B(H)$ by $\Delta|B(H)$.

Stampfli determined the norms of these operators, see [4] for details. For instance, for the norm of inner derivation Δ_A, he showed that

$$\|\Delta_A|B(H)\| = 2d(A) \tag{1}$$

where $d(A) = \inf_{\lambda\in\mathbb{C}} \|A - \lambda\|$.

In order to determine the extent to which the identity (1) applies, L. Fialkow [7] introduced the notion of S - universal operators and went further to give the criteria for S - universality for subnormal operators. We state the following vital definitions;

Definition 2.1 ([3]). *A (symmetric) norm ideal $(\mathfrak{J}, \|.\|_{\mathfrak{J}})$ in $B(H)$ consists of a proper two-sided ideal $\mathfrak{J}$ together with a norm $\|.\|_{\mathfrak{J}}$ satisfying the following conditions;*

- $(\mathfrak{J}, \|.\|_{\mathfrak{J}})$ *is a Banach space.*
- $\|AXB\|_{\mathfrak{J}} \leq \|A\|\|X\|_{\mathfrak{J}}\|B\|$, *for all $X \in \mathfrak{J}$ and all operators A and B in $B(H)$*
- $\|X\|_{\mathfrak{J}} = \|X\|$, *for X a rank one operator.*

For a complete account of the theory of norm ideals, we refer to [11].
Let $(\mathfrak{J}, \|.\|_{\mathfrak{J}})$ be a norm ideal and let $A \in B(H)$. If $X \in \mathfrak{J}$, then $\Delta_A(X) \in \mathfrak{J}$ and $\|AX - XA\|_{\mathfrak{J}} = \|(A-\lambda)X - X(A-\lambda)\|_{\mathfrak{J}} \leq 2\|A-\lambda\|\|X\|_{\mathfrak{J}}$ for all $\lambda \in \mathbb{C}$. Hence $\|\Delta_A(X)\|_{\mathfrak{J}} \leq 2d(A)\|X\|_{\mathfrak{J}}$, implying that $\|\Delta_A|\mathfrak{J}\| \leq 2d(A)$.

Definition 2.2 ([7]). *An operator $A \in B(H)$ is S - universal if $\|\Delta_A|\mathfrak{J}\| = 2d(A)$ for each norm ideal $\mathfrak{J}$ in $B(H)$.*

Definition 2.3 ([4]). *The maximal numerical range of A, $W_o(A) = \{\lambda \in \mathbb{C} : \exists\{x_n\} \subseteq H, \|x_n\| = 1, \lim_n\langle Ax_n, x_n\rangle = \lambda\,, \lim_n \|Ax_n\| = \|A\|\}$ and its normalized maximal numerical range is given by*

$$W_N(A) = \begin{cases} W_0(A/\|A\|), & if \quad A \neq 0, \\ 0, & if \quad A = 0. \end{cases}$$

The set $W_0(A)$ is nonempty, closed, convex and contained in the closure of the numerical range, see [4]. The following elementary results will be useful and are therefore provided here for completion,

Lemma 2.4. *For any $A, B \in B(H)$,*

(i) $\Delta_{A,B}(X) = \Delta_{A-\lambda,B-\lambda}(X)$, *for $X \in B(H)$ and $\lambda \in \mathbb{C}$, $\lambda \neq 0$.*

(ii) $\|\Delta_{A,B}|B(H)\| \leq \|A\| + \|B\|$.

Proof. **(i)** Clear.

(ii) By definition, $\|\Delta_{A,B}|B(H)\| = \sup\{\|\Delta_{A,B}(X)\| : X \in B(H), \|X\| = 1\}$.
But $\|\Delta_{A,B}(X)\| = \|AX - XB\| \leq \|AX\| + \|XB\| \leq \|A\|\|\|X\| + \|X\|\|B\|$.
That is $\|\Delta_{A,B}(X)\| \leq \|A\|\|X\| + \|X\|\|B\|$.
So it immediately follows that $\|\Delta_{A,B}|B(H)\| \leq \|A\| + \|B\|$.

□

We note that when $A = B$, then $\Delta_{A,A} = \Delta_A$ and $\|\Delta_A|B(H)\| \leq 2\|A\|$.
We shall relate the norm of a generalized derivation $\|\Delta_{A,B}|B(H)\|$ to the norms of inner derivations $\|\Delta_A|B(H)\|$ and $\|\Delta_B|B(H)\|$ by restricting ourselves to S - universal operators.

3 Norms Of Derivations

We begin this section by stating the following result from literature,

Theorem 3.1. *If* $A \in B(H)$ *is* S *- universal, then* $\|\Delta_A|B(H)\| = 2\|A\|$.

See [5] for the proof.

Our first result is the following,

Theorem 3.2. *Let* $A, B \in B(H)$ *be* S *- universal. Then*

$$\|\Delta_{A,B}|B(H)\| \leq \frac{1}{2}\Big(\|\Delta_A|B(H)\| + \|\Delta_B|B(H)\|\Big).$$

Proof. From Lemma 2.4 above, we have that for any $A, B \in B(H)$,

$$\|\Delta_{A,B}|B(H)\| \leq \|A\| + \|B\|. \tag{2}$$

But since A, B are S - universal, it follows from Theorem 3.1 that $\|\Delta_A|B(H)\| = 2\|A\|$ and $\|\Delta_B|B(H)\| = 2\|B\|$.
Thus

$$\|\Delta_{A,B}|B(H)\| \leq \frac{1}{2}\Big(\|\Delta_A|B(H)\| + \|\Delta_B|B(H)\|\Big). \tag{3}$$

This completes our proof. □

For any $\lambda \in \mathbb{C}$, the following inequalities are immediate consequences of Theorem 3.2 and their proofs are trivial, hence not included here.

Corollary 3.3. *Let* $A, B \in B(H)$ *be* S *- universal. Then*

(i)

$$\|\Delta_{A,B}|B(H)\| \leq \frac{1}{2}\Big(\|\Delta_{A-\lambda}|B(H)\| + \|\Delta_{B-\lambda}|B(H)\|\Big).$$

(ii)

$$\|\Delta_{A-\lambda,B-\lambda}|B(H)\| \leq \frac{1}{2}\Big(\|\Delta_{A-\lambda}|B(H)\| + \|\Delta_{B-\lambda}|B(H)\|\Big).$$

(iii)

$$\|\Delta_{A-\lambda,B-\lambda}|B(H)\| \leq \frac{1}{2}\Big(\|\Delta_A|B(H)\| + \|\Delta_B|B(H)\|\Big).$$

(iv)

$$\|\Delta_{A,B}|B(H)\| \;\leq\; \frac{1}{2}\Big(\|\Delta_A|B(H)\| + \|\Delta_{B-\lambda}|B(H)\|\Big)$$

and

$$\|\Delta_{A,B}|B(H)\| \;\leq\; \frac{1}{2}\Big(\|\Delta_{A-\lambda}|B(H)\| + \|\Delta_B|B(H)\|\Big).$$

(v)

$$\|\Delta_{A-\lambda,B}|B(H)\| \;\leq\; \frac{1}{2}\Big(\|\Delta_A|B(H)\| + \|\Delta_B|B(H)\| + 2|\lambda|\Big)$$

and

$$\|\Delta_{A,B-\lambda}|B(H)\| \;\leq\; \frac{1}{2}\Big(\|\Delta_A|B(H)\| + \|\Delta_B|B(H)\| + 2|\lambda|\Big).$$

(vi)

$$\|\Delta_{A-\lambda,B}|B(H)\| \;\leq\; \frac{1}{2}\Big(\|\Delta_{A-\lambda}|B(H)\| + \|\Delta_{B-\lambda}|B(H)\| + 2|\lambda|\Big)$$

and

$$\|\Delta_{A,B-\lambda}|B(H)\| \;\leq\; \frac{1}{2}\Big(\|\Delta_{A-\lambda}|B(H)\| + \|\Delta_{B-\lambda}|B(H)\| + 2|\lambda|\Big).$$

Remark 3.4. *The following question seems natural; When does equality hold in equation (3) above?*

The next Theorem answers the question in Remark 3.4 above in the affirmative,

Theorem 3.5. *Let $A, B \in B(H)$ be non - zero S - universal operators. Then $\|\Delta_{A,B}|B(H)\| = \frac{1}{2}\Big(\|\Delta_A|B(H)\| + \|\Delta_B|B(H)\|\Big)$ if and only if $W_N(A) \bigcap W_N(-B) \neq \emptyset$.*

Proof. Let $\|\Delta_{A,B}|B(H)\| = \frac{1}{2}(\|\Delta_A|B(H)\| + \|\Delta_B|B(H)\|)$. Then since A, B are S - universal, it follows that $\|\Delta_{A,B}|B(H)\| = \|A\| + \|B\|$. Hence by Theorem 7 of [4], it follows that $W_N(A) \bigcap W_N(-B) \neq \emptyset$.
To prove the converse, let $\mu \in W_N(A) \bigcap W_N(-B)$. Then there exists two sequences $\{x_n\}$ and $\{y_n\}$ in H such that $\|x_n\| = \|y_n\| = 1$, $\lim_n \|Ax_n\| = \|A\|$, $\lim_n \|By_n\| = \|B\|$, $\lim_n \langle Ax_n, x_n\rangle = \mu\|A\|$, and $\lim_n \langle By_n, y_n\rangle = -\mu\|B\|$.
Take $\lim_n(\alpha_n - \gamma_n) = \mu\|A - B\|$ for $\alpha_n, \gamma_n \in \mathbb{C}$, and set $Ax_n = \alpha_n x_n + \beta_n u_n$ where $\beta_n \in \mathbb{C}$ with $\|u_n\| = 1$ and $x_n \perp u_n$, $\forall\, n$.

Set also $By_n = \gamma_n y_n - \delta_n v_n$ where $\delta_n \in \mathbb{C}$ with $\|v_n\| = 1$ and $y_n \perp v_n, \ \forall n$. Define rank - 2 operators $V_n = (x_n \otimes x_n - y_n \otimes y_n) \circ P_n$ where P_n is the orthogonal projection onto $[x_n, y_n]$.
Then clearly $\|V_n\| = 1 \ \forall\, n$.
So we obtain $\langle Ax_n, u_n\rangle = \langle \alpha_n x_n + \beta_n u_n, u_n\rangle = \langle \alpha_n x_n, u_n\rangle + \langle \beta_n u_n, u_n\rangle = \alpha_n\langle x_n, u_n\rangle + \beta_n\langle u_n, u_n\rangle = \beta_n, \quad \forall\, n$ and
$\langle By_n, v_n\rangle = \langle \gamma_n y_n - \delta_n v_n, v_n\rangle = \langle \gamma_n y_n, v_n\rangle - \langle \delta_n v_n, v_n\rangle = \gamma_n\langle y_n, v_n\rangle - \delta_n\langle v_n, v_n\rangle = -\delta_n, \quad \forall\, n$. With the above conditions, let us consider the quantity $\|A\| + \|B\|$. Then

$$\begin{aligned}
\|A\| + \|B\| &= \frac{1}{\mu}\lim_n \langle Ax_n, x_n\rangle - \frac{1}{\mu}\lim_n \langle By_n, y_n\rangle \\
&= \frac{1}{\mu}\lim_n \Big(\langle Ax_n, x_n\rangle - \langle By_n, y_n\rangle\Big) \\
&= \frac{1}{\mu}\lim_n \Big(\langle \alpha_n x_n + \beta_n u_n, x_n\rangle - \langle \gamma_n y_n - \delta_n v_n, y_n\rangle\Big) \\
&= \frac{1}{\mu}\lim_n \Big(\langle \alpha_n x_n, x_n\rangle + \langle \beta_n u_n, x_n\rangle - \langle \gamma_n y_n, y_n\rangle + \langle \delta_n v_n, y_n\rangle\Big) \\
&= \frac{1}{\mu}\lim_n \Big(\alpha_n\langle x_n, x_n\rangle + \beta_n\langle u_n, x_n\rangle - \gamma_n\langle y_n, y_n\rangle + \delta_n\langle v_n, y_n\rangle\Big) \\
&= \frac{1}{\mu}\lim_n (\alpha_n - \gamma_n) \\
&= \frac{1}{\mu}.\mu\|A - B\| \\
&= \|A - B\|.
\end{aligned}$$

Now

$$\|A\| + \|B\| = \|A - B\| \geq \|AV_n - V_n B\| \geq \|AV_n x_n - V_n B y_n\|.$$

But $\lim_n \|Ax_n\| = \|A\|$ and $\lim_n \|By_n\| = \|B\|$, we conclude that $\limsup_n \|AV_n x_n - V_n B y_n\| = \|A\| + \|B\|$.
Thus we obtain $\|\Delta_{A,B}|B(H)\| \geq \limsup_n \|AV_n x_n - V_n B y_n\| = \|A\| + \|B\|$.
But since $\|\Delta_{A,B}|B(H)\| \leq \|A\| + \|B\|$, we deduce that

$$\|\Delta_{A,B}|B(H)\| = \|A\| + \|B\|. \tag{4}$$

Since A, B are S - universal, then we have $\|A\| = \frac{1}{2}\|\Delta_A|B(H)\|$ and $\|B\| = \frac{1}{2}\|\Delta_B|B(H)\|$.
So that equation (4) becomes

$$\|\Delta_{A,B}|B(H)\| = \frac{1}{2}\Big(\|\Delta_A|B(H)\| + \|\Delta_B|B(H)\|\Big).$$

This completes our proof. ∎

The following Corollaries follow,

Corollary 3.6. *Let $A \in B(H)$ be non-zero S - universal operator. Then $\|\Delta_A|B(H)\| = 2\|A\|$ if and only if $W_N(A) \bigcap W_N(-A) \neq \emptyset$.*

Proof. This follows easily by replacing B with A in Theorem 3.5 and taking note that $\Delta_{A,A} = \Delta_A$. □

Corollary 3.7. *For a non-zero $A \in B(H)$, the following statements are equivalent,*

(i) *A is S - universal*

(ii) $diam(W(A)) = 2\|A\|$

(iii) $diam(W(A)) = \|\Delta_A|B(H)\|$

(iv) $W_N(A) \bigcap W_N(-A^*) \neq \emptyset$.

where $diam(W(A))$ is the diameter of the numerical range of A, $W(A)$.

Proof. $(i) \Longrightarrow (ii) \Longrightarrow (iii)$ Since A is S - universal, $\|\Delta_A|B(H)\| = 2\|A\|$ and $diam(W(A)) = 2\|A\|$, see [5].
$(iii) \Longrightarrow (iv) \Longrightarrow (i)$ This is immediate by appropriate adjustment in Theorem 3.5 above and Theorem 1 of [10]. We omit the details here. □

Another immediate consequence of Theorem 3.5 is the following result proved in general in [4].

Corollary 3.8. *Let $A, B \in B(H)$ be non-zero S - universal operators. Then $\|\Delta_{A,B}|B(H)\| = \|A\| + \|B\|$ if and only if $W_N(A) \bigcap W_N(-B) \neq \emptyset$.*

Remark 3.9. *The results obtained in this paper are still valid even if we restrict our derivations on norm ideal $\mathfrak{J}$ in the algebra $B(H)$ instead of $B(H)$ itself. The reason for this is quite simple. Just recall that the condition of S - universality guarantees the equality $\|\Delta_A|B(H)\| = \|\Delta_A|\mathfrak{J}\|$. These basic concepts are well documented by the current authors in [5].*

4 Conclusion

In this paper, we have investigated norms of inner and generalized derivations implemented by S - universal operators. We have shown here that for these operators, the norm of a generalized derivation implemented by two operators is less or equal to half the sum of the norms of inner derivations implemented by each operator. We have further provided a necessary and sufficient condition for equality to hold. These results are new and have never been investigated.

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