ANALYTIC SOLUTIONS OF CALL AND PUT OPTIONS OF NONLINEAR BLACK-SCHOLES EQUATION WITH TRANSACTION COSTS AND PRICE SLIPPAGE

BY

COLLINS MUSERA

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DECLARATION

This thesis is my own work and has not been presented for a degree award in any other institution.

COLLINS MUSERA

MSC/MAT/00011/2013

Signature .... Date ....

This thesis has been submitted for examination with our approval as the university supervisors.

Dr. Joseph Eyang'an Esekon

Department of Applied Sciences

Murang'a University College

Signature .... Date ....

Prof. Fredrick Onyango

Department of Statistics and Actuarial Science

Maseno University

Signature .... Date ....

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DEDICATION

To my beloved wife Faith and son Bradley

Abstract

A nonlinear Black-Scholes Partial Differential Equation whose nonlinearity is as a result of transaction costs and a price slippage impact that lead to market illiquidity with feedback effects was studied. Most of the solutions obtained in option pricing especially using nonlinear equations are numerical which gives approximate option values. To get exact option values, analytic solutions for these equations have to be obtained. Analytic solutions to the nonlinear Black-Scholes Partial Differential Equation for pricing call and put options to expiry time are currently unknown. The main purpose of this study was to obtain analytic solutions of European call and put options of a nonlinear Black-Scholes Partial Differential Equation with transaction costs and a price slippage impact. The methodology involved reduction of the equation into a second-order nonlinear Partial Differential Equation. By assumption of a traveling wave profile the equation was further reduced to Ordinary Differential Equations. Solutions to all the transformed equations gave rise to an analytic solution to the nonlinear Black-Scholes equation for a call option. Using the put-call parity relation the put option's value was obtained. The solutions obtained will be used to price put and call options in the presence of transaction costs and a price slippage impact. The solutions may also help in fitting the Black-Scholes option pricing model in the modern option pricing industry since it incorporates real world factors hence significantly contributing to the field of mathematical finance. We, therefore, recommend to hedgers and speculators in derivatives markets to make use of option pricing formulae obtained in this research for accurate option pricing so that they can maximize their profits. In conclusion, further research needs to be done to study the exposure from writing a covered call and the exposure from writing a naked put.

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GLOSSARY OF TERMS AND ACRONYMS

Analytic Solution Solution where the answer is in form of an equation.

Arbitrage Buying and selling simultaneously a security at two different prices and markets.

Asset Any possession which has value in an exchange.

Ask Price Lowest price a seller is willing to sell an asset.

Bid Price Highest price a buyer is willing to pay for an asset.

Bid-ask-spread Amount by which the ask price exceeds the bid price.

Black-Scholes Model Financial market model with an analytic option valuation formula.

BS Abbreviation for Black-Scholes.

BSM Abbreviation for Black-Scholes Model.

BSPDE Abbreviation for Black-Scholes Partial Differential Equation.

Bond Promise to pay interest over a specific term at stated future dates and then pay lump sum at the end of the term.

Bond Position Units of the bond.

Boundary Conditions Conditions on the behaviour of the solution (or its derivative) at the boundary of the domain under consideration.

Boundary Value Problem A combination of a PDE along with suitable boundary conditions.

Call Option A contract where at a prescribed time in future the holder of the option may buy a prescribed asset for a prescribed amount.

Classical Solution A solution that is a smooth function.

Closed-Form Formula Is a mathematical expression that can be evaluated in a finite number of operations.

Competitive Market Market where a trader can buy or sell any quantity of a security without changing its price.

Contingent Claim A financial contract with a random payoff that can be positive or negative.

Cumulative Distribution Function Probability that a variable $X(x)$ will be less than x.

Convexity Is a measure of the curvature or second derivative of how the price of a bond varies with interest rate.

Delta Rate at which the value changes with respect to the asset's price.

Delta Hedging Is the perfect elimination of risk, through exploiting correlation between an option and its underlying asset.

Derivative An instrument whose price is derived from or depends on the price of another asset.

Deterministic Term The dt term in a DE.

Differential Equation An equation that relates the derivatives of a (scalar) function depending on one or more variables.

Diffusion A stochastic process which is the solution to a SDE.

Diffusion Process A solution to a Stochastic Differential Equation.

Drift Coefficient of the deterministic term in a SDE.

Dynamic Hedging A strategy involving rebalancing hedge positions as conditions of the market change.

European Option An option that may be exercised only on expiry.

Exercise To implement the option holder's right to buy (for a call) or sell (for a put) the underlying security.

Exercise Price Amount for which the underlying can be sold (put) or bought (call).

Expected Value of a Variable Average value of the variable obtained when alternative values are wighted by their probabilities.

Expiry Date Time a contract comes to an end.

Financial Market An organized institutional mechanism (structure) for exchanging and creating financial assets.

Financial Mathematics A collection of mathematical techniques that find applications in finance.

Frictionless Market Market with no transaction costs and restrictions on trade.

Gamma Rate of change of delta with respect to the price of the asset.

Generalized Wiener Process A stochastic process where in each short period of time δt, the change in a variable is normally distributed with both mean and variance being proportional to δt.

Geometric Brownian Motion A stochastic process where the logarithm of the underlying asset follows a generalized Wiener process.

Going Short Selling the asset.

Hedge Transfer risk by taking the opposite position in the underlying asset.

Hedge Cost Option value at time $t \in [0, T]$.

Hedger An individual who hedges.

Illiquid Market Is a market in which it is difficult to sell assets because of their expense, lack of interested buyers or some other reason.

Initial Boundary Value Problem The combination of the PDE, the initial conditions, and the boundary conditions.

Itô's Lemma A theorem of stochastic calculus showing that second order partial differential terms of a standard Brownian motion can be considered to be deterministic when integration of the differential terms is done over a non-zero period.

Itô Process A stochastic process: a generalized standard Brownian motion with normally distributed jumps.

Liquidity Any asset can be traded on demand at the market price in arbitrary quantities.

Liquidity Risk Risk that it will be impossible to sell a holding of a particular financial instrument at its theoretical price.

Long Position An option position where a trader has executed at least one option trade.

Market Model Model commonly used by traders.

Markovian Strategy A path-independent strategy.

Money Market Account An investment initially equal to say \$1 and increases at the same short-term risk-free rate of interest prevailing at time t.

Martingale Is a process whose expected future value, conditional on the past, is its current value.

Measure Is a collection of probabilities on the set of all possible outcomes, describing how likely each one is

No-Arbitrage Assumption The assumption that no money can be made out of nothing without risk.

Nonlinear Equation An equation where the power of the dependent variable and each derivative contained in the equation is more than one or the coefficients of the dependent variable and of each derivative are either the dependent variable or its derivative(s).

NSE Abbreviation for Nairobi Securities Exchange.

Ordinary Differential Equation A differential equation that relates the derivatives of a (scalar) function that depends on one variable.

ODE Abbreviation for Ordinary Differential Equations.

Option The right to sell or buy an asset.

Partial Differential Equation A differential equation that relates the derivatives of a (scalar) function that depends on two or more variables.

PDE Abbreviation for Partial Differential Equation.

Payoff Payment.

Pay-off Function The value of the option.

Perfect Hedge Hedge where the risk is eliminated completely.

Perfectly Liquid (Elastic) Market A market where arbitrarily large quantities of a security can be traded without affecting its price.

Portfolio Collection of investments.

Put Option A contract where at a prescribed time in future the holder of the option may sell the underlying asset for a prescribed amount.

Put-Call Parity Is the relationship between the underlying asset and the options.

Risk Degree of uncertainty of a return on an asset.

Risk Free No chance of anything going wrong.

Risk-Free Asset An asset whose return in future is known in advance with certainty.

Risk-Free Rate Rate of interest earned without assuming any risks.

Risk-Neutral Not sensitive to risk.

Risk-Neutral Pricing Pricing of an option assuming the world is risk-neutral.

SDE Abbreviation for Stochastic Differential Equation.

Security Piece of paper proving ownership of an investment.

Self-financing Strategy A strategy which never needs addition of extra cash nor afford withdrawals.

Semimartingale A process which can be decomposed into local martingale and drift terms of finite variation.

Slippage Is the difference between the asset price when the trading decision is taken and the price actually realized by a broker or an algorithmic execution system.

Share Corporation's ownership which represents a piece of the assets and earnings of a corporation.

Short Position Position assumed when traders sell shares they do not own.

Smooth Function A continuously differentiable function.

Solution Any function which gives a true statement when plugged into the equation.

Spot Price Current price.

Spot Rate Annualized nominal interest rate of a pure discount bond.

Stochastic Calculus A *calculus* for random processes.

Stochastic Differential Equation An equation whose terms are a sum of stochastic and deterministic components.

Stochastic Process A collection of random variables.

Stock A corporation‘s ownership represented by shares representing a piece of the assets and earnings of a corporation.

Strike Price See exercise price.

Terminal Value Value at maturity.

Trading Strategy A continuous choice of portfolio which may depend on market movements.

Transaction Costs Costs incurred during trading.

Underlying Asset Basic security such as stock and bond.

Variance A measure of uncertainty of a random variable.

Volatility A measure of uncertainty of the return from an asset.

Wiener Process Basic stochastic process obtained when finer and finer limits of random walks are taken.

With Probability 1 Of an event, having probability one of occurring. This is not quite the same as being guaranteed for sure, as, for example, a normal random variable can take the value zero, but with probability one it will not.

LIST OF NOTATIONS

Lower Case

$a(S,t)$: Function.

$b(S,t)$: Function.

c : Wave speed.

dt : Deterministic component.

dW_t : Infinitesimal Brownian motion.

$e^{-r\tau}$: Discount factor.

f : Function.

h : The function $V(S,T)$.

k : Proportional transaction costs.

n : Integer or positive number.

r : Continuously compounded risk-free interest rate.

S_t : Stochastic process.

S_0 : Initial stock price.

S_t^0 : Fundamental stock price.

t : Time.

x : Transformed spatial variable.

Upper Case

B : Bond.

B_t : Bond's value.

$\mathbb{E}$: Expectation.

$\mathbb{E}^Q$: Expectation under risk-neutral measure.

F, $\hat{F}$: Integrable adapted processes.

K : Exercise (Strike) price.

Le : Leland's number.

M_t, Y_t : A Martingale.

$N(d_j)$: Cumulative distribution function.

$N(0, 1)$: Standardized normal distribution.

$\mathbb{P}$: Probability measure.

$\mathbb{R}$: Set of real numbers.

S : Price of the stock

S_T : Price of the stock at maturity

T : Maturity time.

V^C : Value of a Call option

V^P : Value of a Put option

$V(S_T)$: Payoff function.

$V(x,t)$: Function.

$V(x,\tau)$: Function.

$W_n(t)$: Random walk.

W_t : Brownian motion process/ Weiner process.

X : Random variable.

X_i, Y_i, Z_i : Sequence of random variables.

Z_t : A bounded variation.

Greek Case

α : Measure of price slippage.

δt : Infinitesimal time increment.

Δ : The function V_S.

Δt : Transaction frequency.

μ : Constant drift coefficient.

ν : Function.

ζ : Variable.

ζ_0 : Constant of integration.

ρ : Liquidity parameter.

σ : Constant stock volatility.

$\tilde{\sigma}$: Adjusted volatility.

τ : Transformed time variable.

Π : Value of portfolio.

Φ_t : Markovian trading strategy.

ψ : Constant of integration.

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CHAPTER 1

INTRODUCTION

1.1 Background Information

Modern finance overlaps with many fields of mathematics, in particular, probability theory, linear algebra, calculus, partial differential equations, stochastic calculus, numerical mathematics and programming. Financial mathematics is a collection of mathematical techniques applied in finance. The two main approaches used in financial mathematics to compute the price of options are Partial Differential Equations (PDEs) (where option prices are expressed as solutions to certain differential equations), and probability and stochastic processes (where option prices are expressed as expected values of some random variables). This study was mainly about the (PDE) approach.

The Black-Scholes method of modeling derivatives prices was first introduced in 1973, by the Nobel Prize winners Black and Scholes (1973). Essentially, the Black-Scholes-Merton (BSM) approach shows how the price of an option contract can be determined by using a formula of the underlying asset's price, S, and its volatility, σ, the exercise price, K (price of the underlying asset that the contract stipulates), time to maturity of the contract, T, and the risk-free interest rate, r, prevailing in the market. Its discovery has highly influenced the pricing of options since the 1970's, offering also the possibility of extending the approach to other derivative instruments that have characteristics similar to options.

An option is a financial contract, the value of which is derived from the price of an underlying asset, hence the title of "derivatives" attributable to these types of contracts. The holder of such a contract purchases the right, but not the obligation, to buy (incase of a call) or sell (incase of a put) the asset specified in the option contract - the underlying. The classification of options varies according to their scope and flexibility in the terms of their fulfillment. Hence, a call/ put option's owner has the possibility of buying/ selling the underlying asset at the specified contract price and date, for a specified fee - the option premium. This is the price modeled using the BSM approach.

European options which are studied in this work can be executed only at a maturity date, hence the simplicity in modeling European option prices. The second participating party to the option contract is the option writer, who must satisfy the terms of the agreement to buy or sell the asset, should the option holder decide to exercise his right.

The option is one of the most important financial instruments and its use in speculation and hedging has become wide. Because of the popularity of derivative securities, there was a great need for good and reliable ways to compute their prices. Furthermore, options offer a more predictable assessment of risk and returns. To simplify the work of hedgers and speculators in the market of derivatives, it was essential that option pricing formulae are made available so that opportunities for buying or selling options are fully explored and maximized.

If options can be traded with confidence in the market there will be an increase in the avenues for investment and reduction in investor risks as options give investors the opportunity to exchange risks. It has also been established that derivative markets are among the most affordable and convenient means by which investors can cushion themselves against interest rate fluctuations, volatility in exchange rates and commodity price swings.

In the derivative markets, the price of an option in the future may be known and an investor may position himself/herself and make a more informed investment decision. Ever since the Black-Scholes model was introduced in the field of mathematical finance several individuals have done research on it in order to relax some of the assumptions and make the model fit in the current market. Most research done on nonlinear Black-Scholes Partial Differential Equations has been on numerical solutions, and those that have solved them analytically have concentrated on call options.

From the literature, however, an analytic solution of a call and put options in the presence transaction costs and a price slippage impact considering time to expiry are still unknown. This thesis intended to offer this analytic solutions. In this study we built on the work of Backstein and Howison [3] for modeling illiquid markets for derivative assets.

Options are popular because they are extremely attractive to investors both for speculation and hedging. There is a systematic way to determine how much they are worth and hence they can be bought and sold with some confidence.

1.2 Statement of the Problem

The Black-Scholes model is built on some unrealistic assumptions like the non-existence of transaction costs. When this assumption is relaxed, a nonlinear Black-Scholes Partial Differential Equation is obtained. Most of the work done in option pricing especially using nonlinear equations have mainly been on call options leaving out put options yet put options are also traded in the market. Furthermore, these equations have mainly been solved numerically which gives approximate solutions. To get the exact solutions, the equations have to be solved analytically. Analytic solutions that exist for the nonlinear equations are mainly on call options. However, the expiry time, T, and the transformations $x = \ln(\frac{K}{S})$ and $\tau = T - t$ were not considered in all these solutions which hinder the study of put-call parity relation as it relies on time to expiry $T - t$.

Since the inception of the famous Black-Scholes model and despite the growing interest in the field of mathematical finance, analytic solutions to the nonlinear Black-Scholes Partial Differential Equation for pricing European call and put options in which transaction costs and a price slippage impact have been incorporated up to time of expiry are currently unknown. This study sought to obtain this solutions.

1.3 Objectives of the Study

1.3.1 Main Objective

The main objective of this study was to Obtain Analytic Solutions of European call and put options of a nonlinear Black-Scholes Partial Differential Equation with transaction costs and a price slippage impact.

1.3.2 Specific Objectives

The specific objectives of this research were:

1. To find an analytic solution of the nonlinear Black-Scholes Equation for a call option.
2. Use the Put-Call parity relation to obtain an analytic solution of a put option of the nonlinear Black-Scholes equation.

1.4 Significance of the Study

Options have become extremely popular because they are extremely attractive to investors both for speculation and hedging. There is a systematic way to determine how much they are worth and hence they can be bought and sold with some confidence. They are traded not only for the purpose of speculation and hedging but are actively used as a risk management tool as they are tailor made for exchanging risks between actors on the financial market. Both call and put options are traded in the market, since these derivatives facilitate the transfer of risks, they have a positive impact on the economic system.

A put option protects its owner against the risk that the shares he/she owns drops below the strike price. To manage the risks associated with the underlying security, to protect against fluctuations in value and to profit from periods of inactivity or decline, it is only fair if call and put option's fair value in the market can be determined. This cannot be achieved if there are no formulae for pricing the options. The solutions obtained in this study will be used for pricing call and put options in the presence of transaction costs and a price slippage impact.

Increased risk may cause a loss in the company, therefore derivatives may provide a solution to hedge against the price volatility of the asset. It is hoped that with the availability of option pricing formulae the market will attract more investors and this will spur economic growth. This study can significantly contribute to the growth of the securities exchange market hence economic growth. It is also our hope that the solutions that have been obtained may help in fitting the Black-Scholes option pricing model in the modern options industry since they incorporate real world factors like transaction costs and price slippage.

Furthermore, the price of options is expressed in the form of an expectation of suitably discounted future values or cash flows. Derivatives markets can, therefore, enable an increased access to finance by allocating finances to the most suitable investments.

1.5 Organization of the Thesis

This thesis is organized as follows. Chapter 1 is an introductory chapter which highlights the background information, statement of the problem, objectives of the study and the significance of the study. Chapter 2 gives an explanation of some basic and fundamental concepts in option pricing. Chapter 3 reviews the literature of similar studies and gives a detailed explanation of the Black-Scholes option pricing theory. Chapter 4 gives an explanation of the methodology in obtaining the analytic solutions. Chapter 5 covers the results obtained in the study and Conclusions and recommendations come in Chapter 6.

CHAPTER 2

BASIC CONCEPTS

In this chapter, we give some important definitions and facts that were essential in our study.

Black-Scholes model is a model which describes financial markets and derivative instruments mathematically. This model was used by Black and Scholes to get the Black-Scholes Partial Differential Equation which was solved to give Black-Scholes formulae. The formulae are used in valuing European style options. The Black-Scholes formula gives the current value of an option that is based on the price of the stock following a Stochastic Differential Equation. In order to study option pricing in continuous time, the prices are modeled as continuous time stochastic processes by using diffusion processes and stochastic differential equations.

2.1 Random Walk $W_n(t)$

A random walk is the stochastic process formed by the successive summation of independent, identically distributed random variables. Researchers who work with perturbations of random walks, or with particle systems and other models that use random walks as a basic ingredient, often need more precise information on random walk behavior [20].

Definition 2.1.1 (20) For n a positive integer, define the binomial process $W_n(t)$ to have:

1. $W_n(0) = 0$,

2. Layer spacing is $\frac{1}{n}$,

3. Up and down jumps are equal and of size $\frac{1}{\sqrt{n}}$,

4. Measure $\mathbb{P}$, given by up and down probabilities everywhere are equal to $\frac{1}{2}$.

This means that if $X_1, X_2.....$ is a sequence of independent binomial random variables taking values ± 1 with equal probability, then at the i^{th} step, the value of W_n is defined by:

$$W_n = \left(\frac{i}{n}\right) = W_n\left(\frac{i-1}{n}\right) + \frac{X_i}{\sqrt{n}} \quad for \quad all \quad i \geq 1.$$

Theorem 2.1 *(**Central Limit theorem**) [20] Let the random variables $X_1, ..., X_n$ form a random sample whose size is n from a probability distribution with mean and standard deviation μ and σ respectively. Then for all x*

$$\lim_{n\to\infty} Pr\left(\frac{\sqrt{n}(\bar{x}-\mu)}{\sigma} \leq x\right) = N(x)$$

where $N(x)$ is the cumulative distribution function of x and Pr denotes probability.

Definition 2.1.2 Let X be a random variable and $x \in \mathbb{R}$. If X is continuous then it has the probability density function $f : \mathbb{R} \to [0, \infty)$ which satisfies

$$N(x) = Pr(X \leq x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{\frac{-y^2}{2}} dy.$$

From the central limit theorem, the limit of this binomial distribution is that as n gets larger, the distribution of $W_n(1)$ tends to the standardized normal distribution, i.e

$$\lim_{n \to \infty} W_n \sim N(0, 1).$$

In fact

$$W_n(t) = \sqrt{t} \left(\frac{\sum_{i=1}^{nt} X_i}{\sqrt{nt}} \right).$$

By the central limit theorem, the distribution of the ratio in brackets tends to a standardized normal random variable. This means that the distribution of $W_n(t)$ tends to a normal distribution, i.e, $\lim_{n\to\infty} W_n \sim N(0,1)$ hence, in the limit as $n \to \infty$, the distribution of the random walk $W_n(t)$ converges towards Brownian motion W_t [20].

2.2 Brownian Motion

Definition 2.2.1 [22] A *standard Brownian motion* or a *standard Weiner process* is a stochastic process $\{W_t, t \geqslant 0\}$ with the following properties:

1. $W_0 = 0$,
2. With probability 1 the function $t \to W_t$ is continuous in t,
3. The process $\{W_t, t \geqslant 0\}$ has stationary independent increments,
4. The increments $W_{t+s} - W_s$ has the normal $(0, t)$ distribution,

where t is time and the constant $\sigma > 0$ is the volatility (diffusion coefficient).

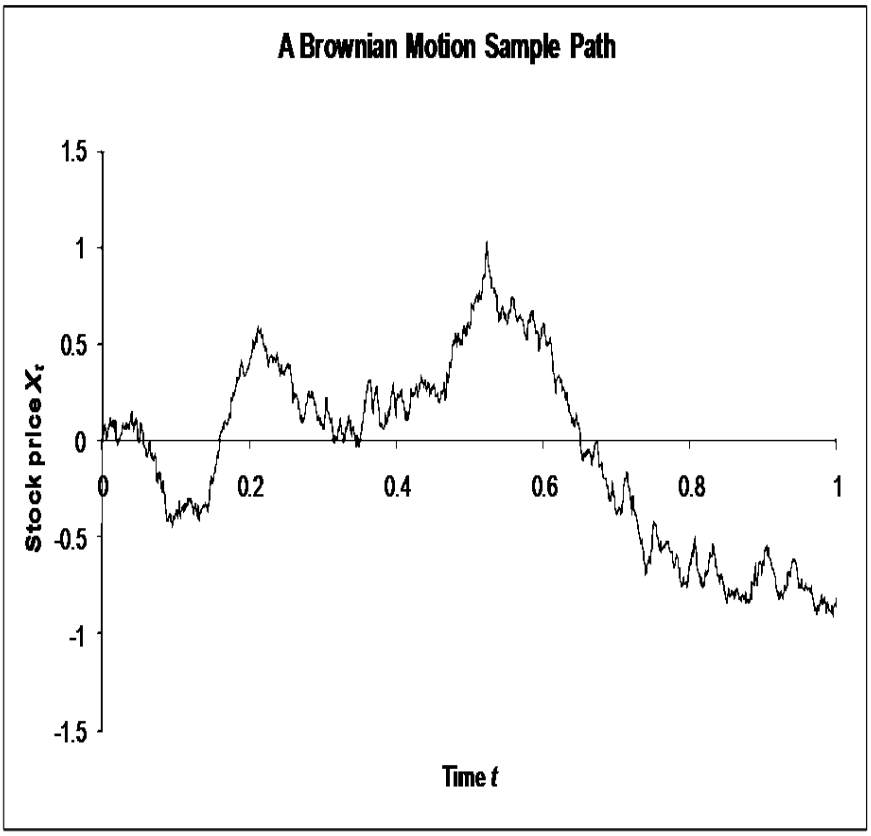

Figure 2.1: A possible realization of a Brownian Motion

2.3 Itô$'s$ Process

[27] Itô$'s$ Process generalize Brownian motion by letting parameters μ and σ to be functions of underlying variable S_t and time t. Thus, a one dimensional generalized Itô process is given in the Itô$'s$ lemma or Itô$'s$ formula stated below.

Lemma 2.3.1 ***(Itô$'s$ Lemma)*** *Suppose that the random variable S is described by the Itô process*

$$dS = a(S,t)dt + b(S,t)dW_t. \tag{2.1}$$

Suppose the random variable $f = V(S,t)$. Then f is described by the following Itô *process:*

$$df = \left(a(S,t)V_S + V_t + \frac{1}{2}\left(b(S,t)\right)^2 V_{SS} \right) dt + b(S,t)V_S dW_t$$

where $a(S,t)$ and $b(S,t)$ are functions.

The Stochastic Differential Equation (2.1) for process, S is a generalized *diffusion (*Itô) *process* [27]. Its integration gives

$$S_t = S_0 + \int_0^t a(S_t,t)dt + \int_0^t b(S_t,t)dW_t$$

The last integral is called a *stochastic* (*Itô*) *integral.* This lemma is the cornerstone of stochastic calculus.

2.4 Geometric Brownian Motion

Definition 2.4.1 [1] A stochastic process S_t is said to follow a *geometric Brownian motion* if it satisfies the following Stochastic Differential Equation:

$$dS_t = \mu S_t dt + \sigma S_t dW_t \tag{2.2}$$

where W_t is a Weiner process, dt is the deterministic component and drift μ, and volatility σ are constants. $\mu S_t dt$ controls the trend and $\sigma S_t dW_t$ controls the random noise. In order to solve for S_t we will apply Itô formula to $d \ln S_t$:

$$
\begin{aligned}
d(\ln S_t) &= \frac{1}{S_t} dS_t + \frac{1}{2}\left(-\frac{1}{S_t^2}\right)(ds_t)^2 \\
d \ln S_t &= \frac{1}{S_t} dS_t - \frac{1}{2}\frac{1}{S_t^2} dS_t^2 \\
&= \frac{1}{S_t} S_t \left[\mu dt + \sigma dW_t\right] - \frac{1}{2}\frac{1}{S_t^2} S_t^2 \left[\sigma^2 dW_t^2\right]
\end{aligned}
$$

Thus, with Probability one $dW_t^2 \to dt$ as $dt \to \infty$

$$
d \ln S_t = \mu dt + \sigma dW_t - \frac{1}{2}\sigma^2 dt. \tag{2.3}
$$

Now, equation (2.3) is integrable using ordinary rules of calculus.

$$
\ln S_t - \ln S_0 = (\mu - \frac{1}{2}\sigma^2)t + \sigma W_t.
$$

$$
\frac{S_t}{S_0} = e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t}.
$$

From this, we get the evolution of the stock price to be

$$
S_t = S_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t}. \tag{2.4}
$$

where $-\frac{1}{2}\sigma^2$ is the modification which is obtained upon use of the It$\hat{o}$ calculus.

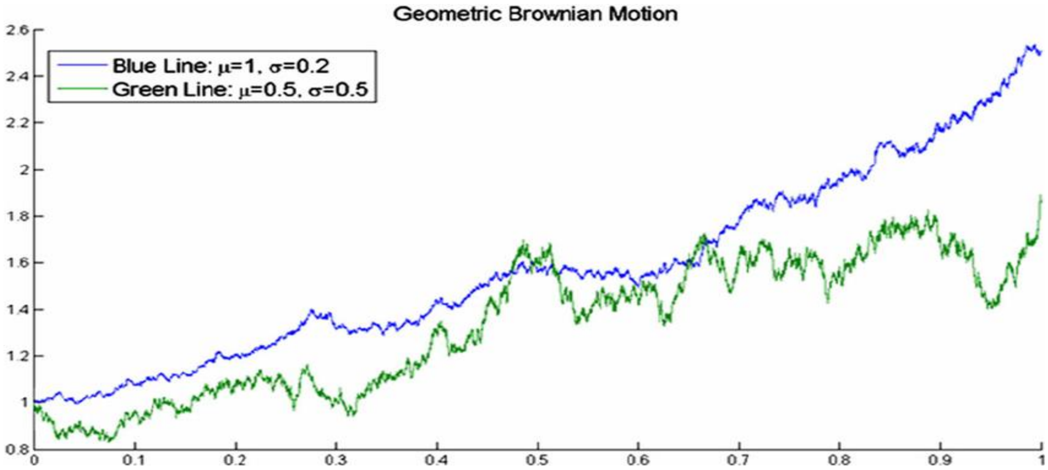

Figure 2.2: A possible realization of a Geometric Brownian Motion

2.5 Martingale

Definition 2.5.1 (15) A *Martingale* is a stochastic process $M_t = \sum_{i=1}^{\infty}\{X_i\}$ such that the unconditional expected value is always finite and that $\mathbb{E}(X_{t+1}|X_t.........., X_1) = X_t$ for all t.

The process M_t is a sub-martingale if $X_t \leq \mathbb{E}(X_{t+1}|X_t.........., X_1)$ for all t. The process M_t is a super-martingale if $X_t \geq \mathbb{E}(X_{t+1}|X_t.........., X_1)$ for all t. The process M_t is a martingale if it is both a super-martingale and a sub-martingale.

It becomes a semi-martingale if it can be written as $M_t = Y_t + Z_t$ where the process $\sum_{i=1}^{\infty}\{Y_i\}$ is a martingale and the process $\sum_{i=1}^{\infty}\{Z_i\}$ is a bounded-variation i.e; $Z_t = F_t - \hat{F}_t$ where $F_1 \leq F_2 \leq$ ...... and $\hat{F}_1 \leq \hat{F}_2$...... are integrable adapted processes [15].

2.6 Markov Process

Definition 2.6.1 (9) A Markov process is a process whereby the distribution of it's future values conditional on it's past and present values depend only on the present and not on the past value. Using the fact that $W_t - W_s$ is independent of W_s, where s is a time variable, we get

$$\begin{aligned}\mathbb{E}(W_t|W_s) &= \mathbb{E}(W_t - W_s|W_s) + \mathbb{E}(W_s|W_s)) \\ &= \mathbb{E}(W_t - W_s) + W_s \\ &= W_s.\end{aligned}$$

We call the expression $\mathbb{E}(W_t|W_s) = W_s$ the martingale property [9].

2.7 Arbitrage–free Pricing

Arbitrage in economics and finance is the practice of taking advantage of differences in prices of an asset between two or more markets. Arbitrage has also been considered as a transaction that involves no negative cash flow at any probabilistic or temporal state and a positive cash flow in at least one state. The profit (risk-free profit) is the difference in the market prices. An *arbitrageur* is an individual engaging in arbitrage. The term arbitrage is commonly applied in trading of financial instruments like bonds, derivatives, currencies and stocks.

If market prices do not allow for profitable arbitrage, they are said to constitute arbitrage-free market or an arbitrage equilibrium. Arbitrage possibilities can arise when any of the following is true:

1. Two assets with equal cash flows are not marketed at the same price.

2. Prices of a particular asset are different on different markets.

3. The asset's price today is not the discount price (at the risk−free interest rate) of the known future price of the asset. Prices in different markets tend to converge to the same price due to the effect of arbitrage [2].

The fundamental theorem of arbitrage/finance is a way to relate arbitrage opportunities with risk neutral measures that are equivalent to the original probability measure. The fundamental theorem of arbitrage-free pricing in a finite state market can be broken down into two parts which state that see [15] for details.

1. There is no arbitrage if and only if a risk-neutral measure equivalent to the original probability measure $\mathbb{P}$ exists, and

2. A market is complete if and only if there is a unique risk-neutral measure equivalent to the original probability measure $\mathbb{P}$.

The fundamental theorem of pricing is, therefore, a way of converting the concept of arbitrage to a question about whether a risk-neutral measure exists or not.

2.8 Price Slippage

The term *price slippage* means trading at unexpected conditions in the market. When a large order has been placed, the large trader will inevitably obtain a worse price for the order than the quoted prices. However, a question that may be asked is whether this large order should have a permanent effect on the price process or not. Assuming that a market maker provides the quotes, and that he has also provided the other side of the large trade, then it is not unreasonable to expect that in the next period the market maker will quote different prices in order to neutralize his position, hence a permanent impact will be felt by the market; this is what Bakstein and Howison [3] term price slippage.

CHAPTER 3

LITERATURE REVIEW

This chapter gives an overview of literature concerning the standard Black-Scholes model and the modified Black-Scholes models.

The Black-Scholes model is one of the most significant concepts in financial mathematics. The theory of option pricing started receiving attention in 1973 when Black and Scholes [4] published their paper. The Partial Differential Equation developed by them and referred to as the Black-Scholes equation, governs the price of the option over time. They developed a closed-form solution to calculate the price of European calls and puts based on certain assumptions by showing how to hedge continuously the exposure on the short position of an option. The crucial idea behind the derivation was to hedge perfectly the option by buying and selling the underlying asset in the right way and eliminate risks.

3.1 Standard Black-Scholes Model

Black and Scholes [4] and Merton [24] in 1973 came up with their famous Black-Scholes model, the first successful model for pricing financial options using the assumptions stated below [17]:

1. The stock price S follows the stochastic process

$$dS = \mu S dt + \sigma S dW$$

with fixed μ and σ.

2. There are no transaction costs associated with hedging of a portfolio.

3. The underlying asset pays no dividends during the life of an option.

4. There are no arbitrage possibilities.

5. Trading of the underlying asset can take place continuously.

6. Short selling is permitted and the assets are divisible.

7. The risk-free interest rate r and the asset volatility σ are known functions of time over the life of the option.

Using these assumptions, Black and Scholes [4] derived a linear Partial Differential Equation see equation (3.6). They used equation (2.2) to model the random behavior of the stock.

Since its inception in the field of mathematical finance, most of the recent research done on it has been to relax the assumptions made in the model. One of the assumptions in the model that was key to this research is that there are no transaction costs, but we all know that trading costs money and when buying and selling securities in the market, one has to pay all the agents and the intermediaries involved in the transaction.

3.1.1 Black-Scholes Partial Differential Equation

This is adapted from [17]. Suppose that we have an option whose value $V(S,t)$ depends on S and t. Let Π denote the value of a portfolio with a long position in the option and a short position in some quantity Δ of the underlying asset. The value of this portfolio is given by:

$$\Pi = V(S,t) - \Delta S. \tag{3.1}$$

By the assumption that the price S of the underlying follows a stochastic process, we get equation (2.2). As time changes from t to $t+dt$, the change in the value of the portfolio is due to the change in the value of the option and the change in the price of the underlying asset. Here Δ is held fixed during the time-step dt. Hence

$$d\Pi = dV - \Delta dS. \tag{3.2}$$

$It\hat{o}'s$ lemma is the most important result of the manipulation of random variables that we require.

The $It\hat{o}'s$ lemma relates the small change in a function of a random variable to a small change in the variable itself and can be applied to functions of any random variable described by a Stochastic Differential Equation. By $It\hat{o}'s$ Lemma, we have

$$dV = \left[V_t + \frac{1}{2}\sigma^2 S^2 V_{SS}\right] dt + V_S dS. \tag{3.3}$$

Combining equations (3.2) and (3.3) yields

$$d\Pi = \left[V_t + \frac{1}{2}\sigma^2 S^2 V_{SS}\right] dt + (V_S - \Delta)dS.$$

Applying delta hedging strategy, we choose $\Delta = V_S$ and this yields

$$d\Pi = \left[V_t + \frac{1}{2}\sigma^2 S^2 V_{SS}\right] dt. \tag{3.4}$$

By the assumption of an arbitrage-free market, the change $d\Pi$ equals the growth of Π in a risk-free interest-bearing account. Hence from equation (3.1) we have

$$d\Pi = r\Pi dt = r(V - \Delta S)dt.$$

From the equation above and equation (3.4) we have

$$r(V - \Delta S)dt = \left[V_t + \frac{1}{2}\sigma^2 S^2 V_{SS}\right] dt. \tag{3.5}$$

Substituting $\Delta = V_S$ and simplifying we arrive at the Black-Scholes equation,

$$V_t + \frac{1}{2}\sigma^2 S^2 V_{SS} + rSV_S - rV = 0 \tag{3.6}$$

for $0 \leqslant t \leqslant T$ where T is the expiration date of the underlying asset. This equation plays a pivotal role in option pricing.

3.1.2 Black-Scholes Formula

Probability theory, Lebesgue integration and Itô$'s$ calculus are the main ingredients in the Black-Scholes formula. The Black-Scholes formula for European options is the exact solution of the BSPDE in equation (3.6). The explicit solution for the European call is

$$C(S,t) = SN(d_1) - Ke^{-r(T-t)}N(d_2) \tag{3.7}$$

where $N(.)$ is the cumulative distribution function for a standardized normal random variable. Here

$$d_1 = \frac{\log \frac{S}{K} + (r + \frac{1}{2}\sigma^2)(T - t)}{\sigma\sqrt{T - t}}$$

and

$$\begin{aligned} d_2 &= \frac{\log \frac{S}{K} + (r - \frac{1}{2}\sigma^2)(T - t)}{\sigma\sqrt{T - t}} \\ &= d_1 - \sigma\sqrt{T - t} \end{aligned}$$

where K is the strike price.

Definition 3.1.1 (European Call option).[17] A European call option is a contract that gives the holder of the option the right but not the obligation to buy an underlying asset for a pre-determined price K on the maturity date T. If the price of the underlying asset at maturity time is less than or equal to the strike price, the option will not be exercised and expires worthless. If the price of the underlying asset at maturity time is greater than the strike price the option may be exercised. The payoff of a Call option is therefore given by

$$V(S_T) = \begin{cases} S_T - K & \text{if } S_T > K; \\ 0 & \text{if } S_T \leq K. \end{cases}$$

where S_T is the price of the underlying at maturity.

Definition 3.1.2 (European Put option).[17] A European put option is a contract that gives the holder of the option the right but not the obligation to sell an underlying asset at a pre-determined strike price K on the maturity date T. If the price of the underlying is less than the strike price at maturity the option may be exercised and if the price of the underlying asset is greater than or equal to the strike price at maturity the option will not be exercised and expires worthless. The payoff of a put option is therefore given by

$$V(S_T) = \begin{cases} K - S_T & \text{if } S_T < K; \\ 0 & \text{if } S_T \geq K. \end{cases}$$

where S_T is the price of the underlying at maturity.

Definition 3.1.3 (Put-Call parity). [17] A Put-Call parity is an important principle in option pricing. It was first identified by economist Hans Stoll in his 1969 paper [8]. He defines the relationship that must exist between European put and call options with the same expiration date and strike price. The relation states that the premium of a call option implies a certain fair price for the corresponding put option having the same strike price K and expiration date T and vice versa. The put-call parity relationship is stated as

$$V^C + Ke^{-r(T-t)} = V^P + S \tag{3.8}$$

where V^C is the current call option value, V^P is the current put option value, $e^{-r(T-t)}$ is the discount factor, K is the strike price and S is the price of the stock.

Using this relation and (3.7), the solution for a put option is given by

$$V^P = Ke^{-r(T-t)}N(-d_2) - SN(-d_1) \tag{3.9}$$

3.2 Modified Black-Scholes Model

All the assumptions (see section 3.1) leading to the Standard Black-Scholes model simplify the calculations but do not necessarily hold in reality.

When some of these assumptions are ignored, a nonlinear Black-Scholes equation is obtained. Nonlinear Black-Scholes equations have been increasingly attracting interest since they provide more accurate values by taking into account more realistic assumptions, such as transaction costs, illiquid markets, risks from the unprotected portfolio or large investor's preferences, which may have an impact on the stock price, the volatility, the drift and the option itself. In this study, the second assumption was ignored to examine how to account for transaction costs for better option pricing.

In the standard Black-Scholes model of pricing options, the influence of transaction costs was neglected. It does not take into account risks from unprotected (non-hedged) portfolio and other effects such as feedback effects on the asset price in the presence of a dominant investor and utility function effect of investors preferences and it was possible to construct a risk-less portfolio that perfectly replicates the option pay-off. However, market practitioners and academia are aware of the importance of transaction costs when considering an investment.

The BSM is based on the assumption of frictionless and perfectly liquid markets which is unrealistic given the scale of hedging activities on many financial markets. When transaction costs are taken into account, perfect replication of the contingent claim is no longer possible. This is what has made it possible in recent years for a number of approaches in dealing with market illiquidity in the pricing and hedging of derivative securities to be developed.

Existing work that has led to such nonlinear Black-Scholes equations to date includes among others Leland [7] who came up with an approximate option replication model featuring transaction costs. The model allows transactions only at discrete times. By a formal delta-hedging argument, Leland derived the nonlinear Black-Scholes equation given by:

$$V_t + \frac{1}{2}\hat{\sigma}^2 S^2 V_{SS} + rV_S - rV = 0 \tag{3.10}$$

where $V_S = \frac{\partial V}{\partial S}$, $V_t = \frac{\partial V}{\partial t}$, $V_{SS} = \frac{\partial^2 V}{\partial S^2}$, S is the price of the stock, t is time, r is the risk-free interest rate and that V is the option price. $\hat{\sigma}$ is an adjusted volatility defined by:

$$\hat{\sigma} = \sigma\sqrt{1 + \frac{2}{\pi} \times \frac{k}{\sigma\sqrt{\Delta t}}}$$

where σ is the original volatility, k is the proportional transaction cost, π is a constant, Δt is the transaction frequency and $\sqrt{1 + \frac{2}{\pi} \times \frac{k}{\sigma\sqrt{\Delta t}}}$ is the Leland's number (Le). Leland's model was partly criticized by Kabanov and Safarian [28] who proved that Leland's result had a hedging error. The restriction of the model was that the convexity of the resulting option price V(hence $V_{SS} > 0$) and the possibility to only consider one option (call) in the portfolio.

Hoggard *et al* [16] studied the nonlinear Black-Scholes equation with the modified volatility for several underlying assumptions and came up with the following model for pricing options with transaction costs given by:

$$V_t + \frac{1}{2}\sigma^2 S^2 V_{SS} + rSV_S - rV - k\sigma S^2\sqrt{\frac{2}{\pi\delta t}}|V_{SS}| = 0 \tag{3.11}$$

where δt is infinitesimal time increment for European call options with strike price K whose analytic solutions for both call and put options are yet to be obtained. However, when transaction costs are large, Hoggard method cannot be used. To deal with this case, Avellaneda and Paras [19] introduced nonlinear obstacle problem.

From a binomial model in [19], making use of the algorithm of Bensaid *et al*, Avellaneda and Paras gave an extension to Hoggard *et al* to general pay-off. Their approach was to divide the range of terminal stock prices into intervals such that the payoff function restricted to each interval is convex and then solve several obstacle problems. They derived the same volatility as Leland's. Dropping the convexity assumption of the resulting option price they stated that incase $V_{SS} \leq 0$ and $Le \geq 1$ (hence $\sigma \leq 0$) the nonlinear equation becomes mathematically ill-posed and does not possess a solution for general pay-off functions. For the case of $V_{SS} > 0$ and $Le \geq 1$ (hence $\sigma > 0$) they proposed several hedging strategies. However, their obstacle problem is set up in such a way that the seller's price in certain cases has to be unreasonably high.

Using the central limit theorem, Boyle and Vorst [23] derived the price of the option from the Binomial model with transaction costs and discrete trading processes, as the time step δt and the transaction cost k tend to zero. The price converges to a Black-Scholes price with modified volatility as in Leland's model with Δt as the mean time length for a change in the value of the stock, not the transaction frequency. They also assumed convexity of the option price V.

Hodges and Neuberger [25] suggested a different approach to model transaction costs. They considered a utility function without specifying it and assumed that the behavior of the investor was characterized by this function. They showed that the Black-Scholes price was the difference between the maximum utility from the final wealth with and without option liability. They postulated that the price of the option in a market with transaction costs should be equal to the unique cash increment which affected that difference. This theory in the presence of transaction costs was further developed by Davis *et al* [21] who derived a model whose price depends on the initial wealth of the investor, the chosen portfolio, the type of utility function, and on the mean return rate of the stock.

Constantinides and Zariphopoulou [6] modified this original definition of the price and obtained universal bounds independent of the utility function. Barles and Soner [5] derived a more complicated model by following the utility function approach of Hodges and Neuberger [25] and considered two optimization problems to come up with the nonlinear Black-Scholes equation given by:

$$V_t + rSV_S + \frac{1}{2}\sigma^2 S^2 V_{SS}[1 + A(e^{r(T-t)}a^2 S^2 V_{SS})] - rV = 0 \tag{3.12}$$

where $A(.)$ is a nonlinear function, r is the constant interest rate, σ is the constant volatility and a is a parameter given by:

$$a = \frac{k}{\sqrt{\epsilon}} = k\sqrt{\gamma N}$$

where γ is the risk aversion factor, N is the number of options to be sold, k is the proportional transaction costs and ϵ is a parameter for risk aversion. V is equal to Black-Scholes price with variable volatility given by

$$\sigma(S,t) = \sigma[1 + A(e^{r(T-t)}a^2 S^2 V_{SS}(S, t : a))]^{\frac{1}{2}}.$$

The volatility depends on the second derivative of the price V_{SS}. The choice of the parameter a depended on how much risk one was willing to take.

Cetin *et al* [26] put forward the predominant model in transaction cost model and came up with a nonlinear Black-Scholes model for pricing European options given by:

$$V_t + \frac{1}{2}\sigma^2 S^2 V_{SS}(1 + 2\rho S V_{SS}) + rSV_S - rV = 0 \qquad (3.13)$$

where ρ is a measure of liquidity. Esekon *et al* [14] considered the (quadratic) transaction-cost model for modeling illiquid markets which is the equation above but with $r = 0$. They solved the nonlinear Black-Scholes equation analytically but for a call option only.

Backstein and Howison in [3] considered the continuous-time feedback effects and came up with an equation for illiquid markets given by:

$$V_t + \frac{1}{2}\sigma^2 S^2 V_{SS}(1 + 2\rho S V_{SS}) + \frac{1}{2}\rho^2(1-\alpha)^2\sigma^2 S^4 V_{SS}^3 + rSV_S - rV = 0 \qquad (3.14)$$

where S is the price of the stock, $\rho \geq 0$ is a measure of the liquidity of the market, σ is volatility, $V(S, t)$ is the option price, and α is a measure of the price slippage impact of a trade felt by all participants of a market for $0 \leq \alpha < 1$ and $1 < \alpha \leq 2$ (see [11]). Backstein and Howison only developed the model and they did not solve the equation analytically.

Esekon [11] obtained an analytic solution for (3.14) but for a European call option only. In obtaining his solution, Esekon did not consider the time to expiry $T - t$ but considered only the current time t. In a more recent research by Esekon [10], the price of a put option for (3.13) was obtained using put-call parity relation but in the absence of price slippage. In all these solutions the transformations $\tau = T - t$ and $x = \ln(\frac{K}{S})$, were not used. This research was an attempt to obtain analytic solutions to the nonlinear Black-Scholes equation for a Call and Put option in which transaction costs and a price slippage impact have been incorporated in relation to the nonlinear Black-Scholes PDE used for modeling illiquid markets that was developed by Backstein and Howison [3] using the transformations mentioned above.

3.3 Backstein and Howison Equation

Considering the continuous-time feedback effects for illiquid markets, Backstein and Howison [3] used two assets in the model: a bond (or a risk-free money market account with spot rate of interest $r \geq 0$) whose value at time t is $B_t \equiv 1$, and a stock S. The stock was assumed to be risky and illiquid while the bond was assumed to be riskless and liquid. The equation (as in Theorem 3.1 of [3]) is given by (3.14). The solution for a Call option for equation (3.14) for time t is found in Theorem 3.2 of [11]. This solution does not consider the value of the option up to time to expiry T.

Liquidity in equation (3.14) above has been defined through a combination of transaction cost and a price slippage impact. Due to ρ, bid-ask spreads dominate the price elasticity effect. When $\alpha = 1$, it corresponds to no slippage and the equation reduces to the PDE given by (3.13) whose solution for a Call option is found in Theorem 3.0.2 of [12] for $r > 0$ and Theorem 4.1 of [14] and in Theorem 3.2 of [13] for $r = 0$. The magnitude of the market impact is determined by ρS. Large ρ implies a big market impact of hedging. If $\rho = 0$, the asset's price in equation (3.14) follows the standard Black-Scholes model in

[4] with constant volatility σ. The boundary conditions for the option are as follows:

$$V(0,t) = 0 \quad for \quad 0 \leq t \leq T$$

,

$$V(S,t) \sim S - Ke^{-r(T-t)} \quad as \quad S \to \infty.$$

We take the last condition to mean that

$$\lim_{S\to\infty} \frac{V(S,t)}{S - Ke^{-r(T-t)}} = 1$$

uniformly for $0 \leq t \leq T$ with the constraint $V(S,t) \geq 0$. In order to price an option, one has to complete several steps:

1. Specify a suitable mathematical model describing sufficiently well the behavior of the stock market;
2. Calibrate the model to available market data;
3. Derive formula or equation for the the price of the option of interest;
4. Compute the price of the option.

Backstein and Howison [3] only developed the model but did no solve it and cannot be used in computing option values as it is. This research obtained analytic solutions for both call and put options so that the model can now be used for pricing this options.

CHAPTER 4

RESEARCH METHODOLOGY

In this study, the nonlinear versions of the European options only were considered. The nonlinearity was due to transaction costs and a price slippage impact that lead to market illiquidity with feedback effects.

To obtain the analytic solution for the options, the transformations $x = \ln(\frac{K}{S})$, and $\tau = T - t$ were applied to the Black-Scholes Partial Differential Equation for modeling illiquid markets in the presence of transaction costs and a price slippage impact.

Assuming a travelling wave solution to the resulting second-order nonlinear Partial Differential Equation reduced it further to Ordinary Differential Equations. All the transformed equations were solved to obtain an analytic solution to the nonlinear Black-Scholes Partial Differential Equation which is used for pricing a call option at $t \geq 0$. Using the put-call parity relation yielded the value of the put option.

The justification of the transformation $x = \ln(\frac{K}{S})$ is that since it can be written as $S = Ke^{-x}$ and brings the notion of discounting. The justification of the transformation $\tau = T - t$ is in order to allow the use of put-call parity relation as put-call relies on time to expiry.

CHAPTER 5

RESULTS

5.1 Introduction

Two primary assumptions are used in formulating classical arbitrage pricing theory: *frictionless* and *competitive* markets. In a frictionless market, there are no transaction costs and restrictions on trade. Restrictions on trade are imposed when we have extreme market conditions. In particular, short sales/purchases are not permitted when the market has shortage/surplus. A trader can buy or sell any quantity of a security without changing its price in a competitive market. The notion of liquidity risk is introduced on relaxing the assumptions above. The purpose of this study was to obtain analytic solutions of the nonlinear Black-Scholes equation arising from transaction costs in the presence of price slippage impact by Backstein and Howison in [3]. This was done using the transformations $x = \ln(\frac{K}{S})$ and $\tau = T - t$.

5.2 Smooth Solution to the Backstein and Howison Equation

5.2.1 Current Value of a Call option

Lemma 5.1 *If $\nu(\zeta)$ is a twice continuously differentiable function, and x and τ are the spatial and time variables respectively, then there exists a travelling wave solution to the* *equation,*

$$V_\tau - \frac{1}{2}\sigma^2(V_{xx} - V_x)(1 + 2(V_{xx} - V_x)) - \frac{1}{2}(1-\alpha)^2\sigma^2(V_{xx} - V_x)^3 + rV_x = 0 \qquad (5.1)$$

in $\mathbb{R} \times [0, \infty)$ *of the form*

$$V(x,\tau) = \nu(\zeta) \quad \text{where} \quad \zeta = x - c\tau, \quad \zeta \in \mathbb{R}; \qquad (5.2)$$

for $0 \le \alpha < 1, \quad 1 < \alpha \le 2, \quad r, \sigma > 0, \quad \tau \ge 0$ *and* $x \in \mathbb{R}$ *such that* $V(x,\tau)$ *is a travelling wave of permanent form which translates to the right with constant speed* $c > 0$.

Proof. Partial differentiation of (5.2) gives

$$V_\tau = -c\nu'(\zeta), \quad V_x = \nu'(\zeta), \quad \textit{and} \quad V_{xx} = \nu''(\zeta),$$

where the prime denotes $\frac{d}{d\zeta}$. Substituting these expressions into (5.1) and rearranging, we conclude that $\nu(\zeta)$ must satisfy the nonlinear second order ODE

$$c\nu' + \frac{1}{2}\sigma^2(\nu'' - \nu')(1 + 2(\nu'' - \nu')) + \frac{1}{2}(1-\alpha)^2\sigma^2(\nu'' - \nu')^3 - r\nu' = 0 \qquad (5.3)$$

in $\mathbb{R}$ and hence $V(x,\tau)$ solves (5.1) as required.

By setting $c = r$ because this is a convection-diffusion equation, the equation resulting from (5.3) can now be solved in a closed-form by first writing it as

$$(1-\alpha)^2(\nu'' - \nu')^2 + 2(\nu'' - \nu') + 1 = 0$$

in $\mathbb{R}$ where $(1-\alpha)^2 \neq 0$. The quadratic equation above is solved to get

$$\nu'' - \nu' = \frac{-1 \pm \sqrt{1-(1-\alpha)^2}}{(1-\alpha)^2}, \quad 0 \leq \alpha < 1, \quad 1 < \alpha \leq 2.$$

Upon integration, we get the variable separable standard form (see[19])

$$\nu' = e^{\zeta_0+\zeta} - \frac{-1 \pm \sqrt{1-(1-\alpha)^2}}{(1-\alpha)^2}, \quad 0 \leq \alpha < 1, \quad 1 < \alpha \leq 2, \quad \zeta_0, \zeta \in \mathbb{R}$$

where ζ_0 is a constant of integration. This is the first order linear autonomous and separable ODE whose solution upon integration is given by

$$\nu(\zeta) = e^{\zeta_0+\zeta} - \frac{-1 \pm \sqrt{1-(1-\alpha)^2}}{(1-\alpha)^2}\zeta + \psi$$

for $0 \leq \alpha < 1, \quad 1 < \alpha \leq 2, \quad \zeta_0, \zeta, \psi \in \mathbb{R}$, where ψ is another constant of integration. Applying the initial condition

$$\nu(0) = 0$$

to the equation above and simplifying gives

$$\psi = -e^{\zeta_0}.$$

Hence

$$\nu(\zeta) = e^{\zeta_0+\zeta} - \frac{-1 \pm \sqrt{1-(1-\alpha)^2}}{(1-\alpha)^2}\zeta - e^{\zeta_0}$$

for $0 \leq \alpha < 1, \quad 1 < \alpha \leq 2, \quad \zeta_0, \zeta \in \mathbb{R}$. Since $V(x,\tau) = \nu(\zeta) = x - r\tau = x - c\tau$, we obtain

$$V(x,\tau) = e^{x_0+(x-r\tau)} - \frac{-1 \pm \sqrt{1-(1-\alpha)^2}}{(1-\alpha)^2}(x-r\tau) - e^{x_0}, \tag{5.4}$$

for $0 \leq \alpha < 1, \quad 1 < \alpha \leq 2, \quad x_0, x \in \mathbb{R}, \quad r > 0, \quad \tau \geq 0$ since $\zeta_0 = x_0 - c \cdot 0 = x_0$ and $c = r$.

Theorem 5.2 *If $V(x,\tau)$ is any positive solution to the nonlinear equation*

$$V_\tau - \frac{1}{2}\sigma^2(V_{xx} - V_x)(1 + 2(V_{xx} - V_x)) - \frac{1}{2}(1-\alpha)^2\sigma^2(V_{xx} - V_x)^3 + rV_x = 0$$

in $\mathbb{R} \times [0, \infty)$ *then*

$$V^C = \frac{1}{\rho}\left[-\left(\frac{-1 \pm \sqrt{1-(1-\alpha)^2}}{(1-\alpha)^2}\{\ln(\frac{K}{S}) - r\tau\} + \frac{K}{S_0}\right)S + \frac{K^2}{S_0}e^{-r\tau}\right] \tag{5.5}$$

in $\mathbb{R} \times [0, \infty)$ *solves the nonlinear Black-Scholes equation*

$$V_t + \frac{1}{2}\sigma^2 S^2 V_{SS}(1 + 2\rho S V_{SS}) + \frac{1}{2}\rho^2(1-\alpha)^2\sigma^2 S^4 V_{SS}^3 + rSV_S - rV = 0 \tag{5.6}$$

Proof. To obtain the solution to (5.6) we apply the transformations

$$x = \ln(\frac{K}{S})$$

$$V(x,\tau) = \rho\frac{V(S,t)}{Ke^{-x}}$$

and

$$\tau = T - t$$

to get

$$V_t = -\frac{S}{\rho}V_\tau,$$

$$V_S = \frac{1}{\rho}(V - V_x),$$

$$V_{SS} = \frac{1}{\rho S}(V_{xx} - V_x).$$

Substituting these expressions into (5.6) gives the equation in Theorem (5.2). Hence applying the transformations above into (5.4) gives (5.5).

5.2.2 Put-Call Parity Relation

The put-call parity relation is given by (4.8) where $V^P(S,t)$ is the put option's current value. By this relation

$$V^P = V^C + Ke^{-r(T-t)} - S \tag{5.7}$$

5.2.3 Current Value of a Put option

Corollary 1 *If $V(x,\tau)$ is any positive solution to the nonlinear equation*

$$V_\tau - \frac{1}{2}\sigma^2(V_{xx} - V_x)(1 + 2(V_{xx} - V_x)) - \frac{1}{2}(1-\alpha)^2\sigma^2(V_{xx} - V_x)^3 + rV_x = 0$$

in $\mathbb{R} \times [0,\infty)$ then

$$V^P = \frac{1}{\rho}\left[-\left(\frac{-1 \pm \sqrt{1-(1-\alpha)^2}}{(1-\alpha)^2}\{\ln(\frac{K}{S}) - r\tau\} + \frac{K}{S_0}\right)S + \frac{K^2}{S_0}e^{-r\tau}\right] + Ke^{-r\tau} - S \tag{5.8}$$

in $\mathbb{R} \times [0,\infty)$ solves the nonlinear Black-Scholes equation

$$V_t + \frac{1}{2}\sigma^2 S^2 V_{SS}(1 + 2\rho S V_{SS}) + \frac{1}{2}\rho^2(1-\alpha)^2\sigma^2 S^4 V_{SS}^3 + rSV_S - rV = 0. \tag{5.9}$$

for $r,\ K,\ S,\ S_0,\ \rho > 0,\ \tau \geq 0,\quad 0 \leq \alpha < 1$ *and* $\quad 1 < \alpha \leq 2$, *where* S_0 *is the initial stock price.*

Proof. To obtain the solution to equation (5.9) the procedure is same as in Theorem (5.2) using equation (5.7) and the transformations $x = \ln(\frac{K}{S})$ and $\tau = T - t$.

CHAPTER 6

CONCLUSIONS AND RECOMENDATIONS

The interplay between finance and mathematics is very challenging and often stimulates the use of innovative mathematics and computational mathematics techniques. In this thesis, we have built on the work of Backstein and Howison (2003).

6.1 Conclusions

We have obtained analytic solutions of European call and put options of nonlinear Black-Scholes Equation in the presence of transaction costs and a price slippage impact that lead to market illiquidity with feedback effects. Assuming the solution of a forward wave, classical solutions for the nonlinear Black-Scholes equation were found. The solutions obtained can be used for pricing European call and put options at time $t \geq 0$. After obtaining the value of a call option, put-call parity relation was used to obtain the put option's value at $t \geq 0$.

The solutions obtained in this research supports the comments in [4, 12, 14]. It is clear that the time to expiry is the most critical for option-pricing models. The Bakstein and Howison model [3] admits to well-posed solutions of options on time to expiry because firstly, whilst remaining well-posed close to expiry the option price behavior also remains sufficiently different from that of the corresponding Black-Scholes (liquid) option. Secondly, in the limit of no price slippage, this model reduces to the model of Cetin *et al.*,[26] which has become a popular model for liquidation costs in recent years.

6.2 Recommendations

We, therefore, recommend to hedgers and speculators in derivatives markets to make use of option pricing formulae obtained in this research for accurate option pricing so that they can maximize their profits. As a line of future research, it is very interesting to consider the possibility of using the put-call parity to study the exposure from writing a covered call and the exposure from writing a naked put.

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Appendix A

Stochastic Processes

We used several terms and concepts of probability theory and stochastic processes. In this chapter, we recall some definitions.

A.1 Probability Space

Let Ω be a *sample space* representing all possible scenarios (e.g. all possible paths for the stock price over time). A subset of Ω is an *event* and $\omega \in \Omega$ is a sample point.

Definition A.1.1 Let Ω be a nonempty set and $\mathcal{F}$ be a collection of subsets of Ω. $\mathcal{F}$ is called a $\sigma - algebra$ (not related to the volatility), if

1. $\Omega \in \mathcal{F}$,
2. Whenever a set A belongs to $\mathcal{F}$, its complement A^c also belongs to $\mathcal{F}$ and
3. Whenever a sequence of sets A_n, $n \in \mathbb{N}$ belongs to $\mathcal{F}$, their union $\bigcup_{n=1}^{\infty} A_n$ also belongs to $\mathcal{F}$.

In our financial scenario, $\mathcal{F}$ represents the space of events that are observable in the market and therefore, all the information available until the time t can be regarded as a $\sigma - algebra$ $\mathcal{F}_t$. It is logical $\mathcal{F}_t \subseteq \mathcal{F}_s$ for $t < s$, since the information that has been available t is still available at s.

Definition A.1.2 Let Ω be a nonempty set and $\mathcal{F}$ be a $\sigma - algebra$ of subsets of Ω. A probability measure $\mathbb{P}$ is a function that assigns a number in $[0,\ 1]$ to every set $A \subseteq \mathcal{F}$. The number is called the probability of A and is written $P(A)$. We require:

1. $P(\Omega) = 1$ and
2. Whenever a sequence of disjoint sets A_n, $n \in \mathbb{N}$ belongs to $\mathcal{F}$, then $P\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} P(A_n)$.

The triple $(\Omega, \mathcal{F}, P)$ is called a probability space.

A.2 Random Variables

Definition A.2.1 A real-valued function X on Ω is called a random variable if the sets $\{X \leq x\} := \{\omega \in \Omega : X(\omega) \leq x\} = X^{-1}(] - \infty, x])$ are measurable for all $x \in \mathbb{R}$. That is $\{X \leq x\} \in \mathcal{F}$.

A.3 Geometric Brownian Motion

For a Geometric Brownian motion

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

Assuming the stock follows a log-normal distribution, by introducing $V_t = \ln S_t$ and applying $\text{It}\hat{o}'s$ Lemma we get

$$dV = \frac{1}{S} dS + \frac{1}{2}\left(-\frac{1}{S^2}\right) dS^2$$

$$= \frac{1}{S}\left[\mu S dt + \sigma S dW\right] + \frac{1}{2}\left(-\frac{1}{S^2}\right)\sigma^2 S^2 dt$$

$$= (\mu dt + \sigma dW) - \frac{1}{2}\sigma^2 dt$$

$$dV = \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma dW_t.$$

Appendix B

Payoff Functions

B.1 Call Options

$V(S,T) = (S-K)^+ \quad \text{for } 0 \le S < \infty.$

$V(0,t) = 0 \quad \text{for } 0 \le t \le T.$

$V(S,t) \sim S - Ke^{-r(T-t)} \quad \text{as } S \to \infty.$

B.2 Put Options

$V(S,T) = (K-S)^+ \quad \text{for } 0 \le S < \infty.$

$V(0,t) = Ke^{-r(T-t)} \quad \text{for } 0 \le t \le T.$

$V(S,t) \to 0 \quad \text{as } S \to \infty.$